MA0301 ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2020

ENGLISH VERSION

EXAM

Exercise 1 (Sets):	10 points
Exercise 2 (Logic):	10 points
Exercise 3 (Relations):	10 points
Exercise 4 (Induction):	20 points
Exercise 5 (Functions):	15 points
Exercise 6 (Graphs):	15 points
Exercise 7 (Combinatorics):	8 points
Exercise 8 (Finite state automata & machines)	12 points

Total: 100 points

Note: In each of the multiple choice exercises exactly one answer is correct.

Exercise 1. Sets (10 points)

(1) **(1 point)** *D_n is the set of positive integers which divide exactly the positive integer n. Which of the three statements is correct?*

A) $D_{60} \cap D_{84} = D_{12}$.

B) $D_{60} \cap D_{84} = D_6$.

C) $D_{60} \cap D_{84} = \emptyset$.

(2) **(1 point)** *Which of the three statements is correct?*

I) Let $X := \{x_1, x_2, x_3, x_4, x_5\}$ and $Y := \{y_1, y_2, y_3, y_4\}$. The cardinality of the powerset of the cartesian product of X and Y is larger than $2^{|X||Y|+1}$.

II) Let $X := \{x_1, x_2, x_3, x_4, x_5\}$ and $Y := \{y_1, y_2, y_3, y_4\}$. The cardinality of the powerset of the cartesian product of X and Y is 1048576.

III) Let $X := \{x_1, x_2, x_3, x_4, x_5\}$ and $Y := \{y_1, y_2, y_3, y_4\}$. The cardinality of the powerset of the cartesian product of X and Y is smaller than 1038576.

(3) **(3 points)** *Let X, Y, and Z be sets. Demonstrate that $\overline{X \cap Y \cap Z} = \overline{X} \cup \overline{Y} \cup \overline{Z}$ by showing that each side is a subset of the other side.*

(4) **(5 points)** *Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $P_1 = \{\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8\}\}$ and $P_2 = \{\{1, 2, 3, 4\}, \{5, 7\}, \{6, 8, 9\}\}$ be two partitions of A. Compute the following set*

$$P_3 := \{P_{1i} \cap P_{2j} \mid i = 1, 2,\ j = 1, 2, 3\} \backslash \emptyset,$$

where P_{11}, P_{12} and P_{21}, P_{22}, P_{23} are the blocks of P_1 and P_2, respectively. In words, the set P_3 consists of all intersections between the blocks of P_1 and P_2, excluding the empty set. Show that P_3 is a partition of A.

Date: 9 May 2020.

Exercise 2. Logic (10 points)

(1) **(1 point)** *Which of the three statements is correct?*

<u>A</u>) *The negation of* $\exists x \forall y \ (p(x,y) \wedge \neg q(x,y))$ *is* $\exists x \forall y \ (p(x,y) \vee q(x,y))$.

<u>B</u>) *The negation of* $\exists x \forall y \ (p(x,y) \wedge \neg q(x,y))$ *is* $\forall x \exists y \ (p(x,y) \vee q(x,y))$.

<u>C</u>) *The negation of* $\exists x \forall y \ (p(x,y) \wedge \neg q(x,y))$ *is* $\forall x \exists y \ (p(x,y) \Rightarrow q(x,y))$.

(2) **(2 points)** *Which of the three statements is correct?*

<u>I</u>) *For primitive statements* p *and* q, *the statement* $\neg p \vee (q \Rightarrow (p \wedge q))$ *is a tautology.*

<u>II</u>) *For primitive statements* p *and* q, *the statement* $\neg p \vee (q \Rightarrow (p \wedge q))$ *is a contradiction.*

<u>III</u>) *For primitive statements* p *and* q, *the statement* $\neg p \vee (q \Rightarrow (p \wedge q))$ *is unsatisfiable.*

(3) **(3 points)** *Use a truth table to show that* $p \Rightarrow (q \vee r)$ *is logically equivalent to* $(p \Rightarrow q) \vee (p \Rightarrow r)$.

(4) **(4 points)** *For primitive statements* p, q, r, s, *use the laws of logic to simplify the statement*

$$\big(\neg q \vee (((\neg r \wedge p) \wedge q) \vee (p \wedge (r \wedge q)))\big) \Rightarrow s.$$

Exercise 3. Relations (10 points)

(1) **(2 points)** *Which of the three statements is correct?*

<u>A</u>) *Consider the positive integers together with the relation* $R := \{(n, m) \mid n + m \text{ even}\}$. *Then* R *defines a poset on the positive integers.*

<u>B</u>) *Consider the positive integers together with the relation* $R := \{(n, m) \mid n{+}m \text{ even}\}$. *Then* R *defines an equivalence relation on the positive integers.*

<u>C</u>) *Consider the positive integers together with the relation* $R := \{(n, m) \mid n{+}m \text{ even}\}$. *Then* R *defines a linear order on the positive integers.*

(2) **(3 points)** *Consider the following Hasse diagram*

$$\begin{tikzcd}[column sep=small, row sep=small]
 & a_1 \arrow[dl, dash] \arrow[dr, dash] & & \\
a_3 \arrow[dr, dash] & & a_2 \arrow[dl, dash] \arrow[dr, dash] & \\
 & a_5 & & a_4
\end{tikzcd}$$

corresponding to the poset (A, R). *Write down* A *and* $R \subseteq A \times A$.

(3) **(5 points)** *Determine the minimal elements of the set*

$$S = \{\{1\}, \{2,3\}, \{1,4\}, \{3,4\}, \{3,5\}, \{1,2,4\}, \{2,4,5\}, \{1,2,3,4,5\}\}.$$

with the proper subset order $\subsetneq$.

Exercise 4. Induction (20 points)

(1) **(3 points)** *Use induction to show that for* $m > 0$

$$\sum_{k=1}^{m} 2^{k-1}k^2 = 2^m(m(m-2)+3) - 3.$$

(2) **(4 points)** *Show that if* m *is a positive integer, then the number* $x_m := 8^m - 14m + 27$ *is divisible by* 7.

(3) **(5 points)** *Show that for* $m > 0$

$$\sum_{k=1}^{m} k^3 = \Big(1 + 2 + 3 + \cdots + m\Big)^2.$$

(4) **(8 points)** *Let* $(F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, \ldots)$, $F_n = F_{n-1} + F_{n-2}$, $n > 1$, *be the Fibonacci numbers. Use the identity* $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$, $n > 0$, *to show that for* $m > 0$

$$\sum_{i=1}^{m} F_{4i-2} = F_{2m}F_{2m}.$$

Exercise 5. Functions (15 points)

(1) **(1 point)** *Which of the three statements is correct?*

A) If the functions $f : A \to B$ and $g : B \to C$ are surjective, then the composite $g \circ f : A \to C$ is surjective.

B) If the functions $f : A \to B$ and $g : A \to B$ are surjective, then the composite $g \circ f : A \to B$ is surjective.

C) Show that if the functions $f : A \to B$ and $g : B \to C$ are surjective, then the composite $f \circ g : B \to C$ is surjective.

(2) **(2 points)** *Which of the three statements is correct?*

I) The function $h : \mathbb{R} \to [0,1)$, $t \mapsto h(t) := t - \lfloor t \rfloor$, is injective. Here $[0,1) := \{x \in \mathbb{R} \mid 0 \leq x < 1\}$.

II) The function $h : \mathbb{R} \to [0,1)$, $t \mapsto h(t) := t - \lfloor t \rfloor$, is not surjective. Here $[0,1) := \{x \in \mathbb{R} \mid 0 \leq x < 1\}$.

III) The function $h : \mathbb{R} \to [0,1)$, $t \mapsto h(t) := t - \lfloor t \rfloor$, is not injective. Here $[0,1) := \{x \in \mathbb{R} \mid 0 \leq x < 1\}$.

(3) **(3 points)** *Which of the three statements is correct?*

i) The inverse function of the function $g(x) = \frac{6-4x}{14-10x}$ is $g^{-1}(x) = \frac{7x+3}{5x-2}$.

ii) The inverse function of the function $g(x) = \frac{2x-3}{5x-7}$ is $g^{-1}(x) = \frac{7x-3}{5x+2}$.

iii) The inverse function of the function $g(x) = \frac{3-2x}{7-5x}$ is $g^{-1}(x) = \frac{7x-3}{5x-2}$.

(4) **(4 points)** *Define the function G on the positive integers recursively:*

$$G(1) = 0 \quad \text{and for } n > 1, \qquad G(n) = G(\lfloor n/2 \rfloor) + 1.$$

Compute $G(27)$, $G(26)$ and $G(25)$. Is G an injective function?

(5) **(5 points)** *Consider a function $f : A \to B$ such that $f(A) = B$. Define for $b \in B$ the set $f^{-1}(b) := \{a \in A \mid f(a) = b\} \subseteq A$. Show that $P := \{f^{-1}(b) \mid b \in B\}$ defines a partition of A.*

Exercise 6. Graphs (15 points)

(1) **(1 point)** *Which of the three statements is correct?*

A) Consider a graph $G(V,E)$ with $|E| = 6$. The minimum number of vertices necessary for G to be planar is four.

B) Consider a graph $G(V,E)$ with $|E| = 6$. The minimum number of vertices necessary for G to be planar is five.

C) Consider a graph $G(V,E)$ with $|E| = 6$. The minimum number of vertices necessary for G to be planar is six.

(2) **(1 point)** *Which of the three statements is correct?*

I) Euler's formula says that in a plane drawing of a connected planar graph $G(V,E)$, the numbers of vertices, $|V(G)|$, edges, $|E(G)|$, and regions, $|R(G)|$, of G satisfy: $|V(G)| - |E(G)| + |R(G)| = 2$.

II) Euler's formula says that in a plane drawing of a connected planar graph $G(V,E)$, the numbers of vertices, $|V(G)|$, edges, $|E(G)|$, and regions, $|R(G)|$, of G satisfy: $|V(G)| + |E(G)| + |R(G)| = 2$.

III) Euler's formula says that in a plane drawing of a connected planar graph $G(V,E)$, the numbers of vertices, $|V(G)|$, edges, $|E(G)|$, and regions, $|R(G)|$, of G satisfy: $|V(G)| - |E(G)| - |R(G)| = 2$.

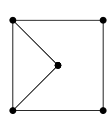
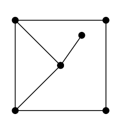
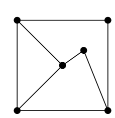
(3) **(2 points)** *Which of the three statements is correct?*

D) Let $G(V,E)$ be a finite connected graph. Its spanning trees have $|V|$ edges.

E) Let $G(V,E)$ be a finite connected graph. Its spanning trees have $|E|$ edges.

F) Let $G(V,E)$ be a finite connected graph. Its spanning trees have $|V| - 1$ edges.

(4) **(2 points)** *Check Euler's formula for the graphs G_a, G_b, G_c below:*

G_a  G_b  G_c 

(5) **(4 points)** *Consider the following matrix*

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

and draw the corresponding directed graph $G(V,E)$.

(6) **(5 points)** *Let $G(V,E)$ be a finite connected planar graph with at least three vertices. Show that the graph $G(V,E)$ has at least one vertex of degree smaller than six.*

Exercise 7. Combinatorics (8 points)

(1) **(1 point)** *Which of the three statements is correct?*

A) The number of permutations of the letters in XAYEXAZZ is 5040.

B) The number of permutations of the letters in XAYEXAZZ is 10080.

C) The number of permutations of the letters in XAYEXAZZ is 40320.

(2) **(2 points)** *Which of the three statements is correct?*

I) The binomial formula says that in the expansion of $(2x+3y^2)^5$ *one finds the term* $280x^4y^2$.

II) The binomial formula says that in the expansion of $(2x+3y^2)^5$ *one finds the term* $220x^4y^2$.

III) The binomial formula says that in the expansion of $(2x+3y^2)^5$ *one finds the term* $240x^4y^2$.

(3) **(5 points)** *Find the number of permutations formed from the letters of the word HEGAHEPP, where the two H's are next to each other.*

Exercise 8. Finite state automata and machines (12 points)

(1) **(1 point)** *Which of the three statements is correct?*

A) Let $\Sigma = \{0,1,2\}$. The word $w = 012$ belongs to $L(r)$ for the regular expression $r = 0^* \vee (1 \vee 2)^*$.

B) Let $\Sigma = \{0,1,2\}$. The word $w = 012$ belongs to $L(r)$ for the regular expression $r = 0^*(1 \vee 2)^*$.

C) Let $\Sigma = \{0,1,2\}$. The word $w = 012$ belongs to $L(r)$ for the regular expression $r = 0^*(1 \vee 2)$.

(2) **(2 points)** *Draw the state diagram of the finite state machine M corresponding to the transition table*

M	ν		ω	
	0	1	0	1
s_0	s_1	s_2	0	1
s_1	s_3	s_1	1	2
s_2	s_1	s_0	2	0
s_3	s_0	s_2	2	0

What is the output corresponding to the input sequence 00101001101*?*

(3) **(4 points)** *What is the language L accepted by the automaton A_1 in Figure 1.*

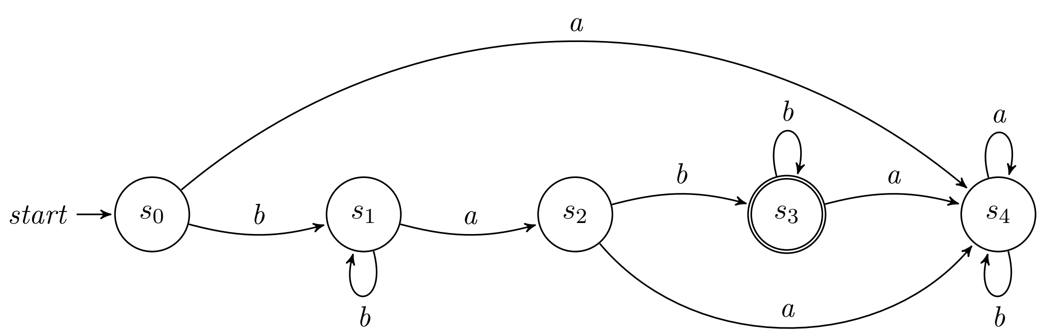


FIGURE 1. The automaton A_1.

(4) **(5 points)** *Let $B = \{a, b\}$. Draw an automaton A_2 with three states, that accepts those words made of letters from B, where the number of b's can be exactly divided by three.*