



Department of Mathematical Sciences

Examination paper for
MA0301 Elementary Discrete Mathematics

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Permitted examination support material: D: No printed or hand-written support material is allowed. A specific basic calculator is allowed.

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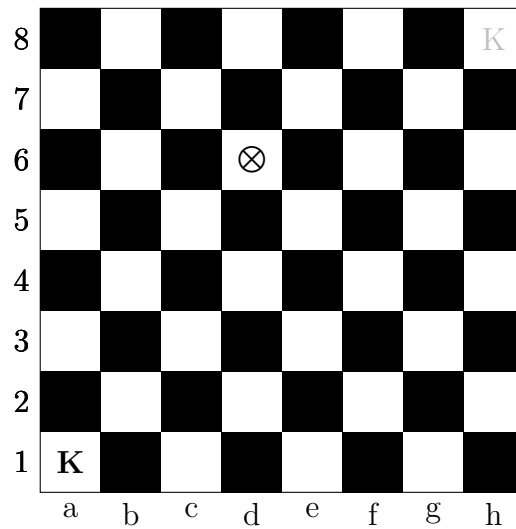
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Date

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Problem 1 [10' + 5']

On a chessboard, there is a king on a1 (bottom left corner). Each time, the king can go up or right a step.



- a) In how many ways can he get to h8 (top right corner)?
- b) In how many ways can he get to h8 (top right corner) without passing through d6?

Problem 2 [10']

$f: \mathbb{R} \rightarrow \mathbb{R}$ is a function. A strict statement for $\lim_{x \rightarrow 0} f(x) = 0$ is

$$\forall \epsilon > 0, \exists \delta > 0 \quad (|x| < \delta \rightarrow |f(x)| < \epsilon).$$

Write the corresponding strict statement for $\lim_{x \rightarrow 0} f(x) \neq 0$.

Note: you do not need to know what is lim to solve the problem.

Problem 3 [10']

Use mathematical induction to prove

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

for any positive integer n .

Note: you have to use induction; other methods do not count.

Problem 4 [5' + 5' + 5']

Are the following functions injective, surjective, bijective or none of them?

a) $f: \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = 2x + 1$.

b) $g: \mathbb{Z} \rightarrow \mathbb{Z}, g(x) = \lfloor \frac{x}{2} \rfloor$.

c) $h = g \circ f$.

Problem 5 [10']

Design a finite state machine as a translator:

- The input strings are strings in $\{0, 1\}^*$ and the output strings are strings in $\{X, Y, Z, W, Q\}^*$. We denote the empty letter by λ .
- After it first encounters the string 111, it starts to translate.
- The translation process is as follows:

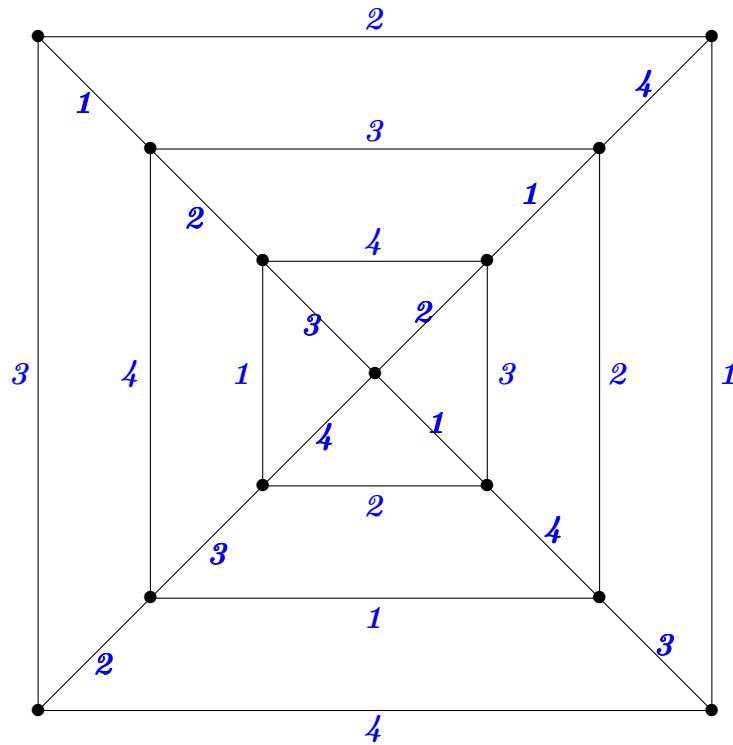
$$\left\{ \begin{array}{ll} 10 & \mapsto X \\ 01 & \mapsto Y \\ 11 & \mapsto Z \\ 001 & \mapsto W \\ 000 & \mapsto Q \end{array} \right.$$

- It will stop translating after outputting a letter Q for the first time.

Note that this is NOT a sequence recognizer. So the machine will not necessarily stop translating when it first encounters a 000. For instance, if the input string is 111 11 10 001 000 010, the output should be $ZXWQ$.

Problem 6 [5' + 5' + 5' + 5']

Consider the following unoriented weighted graph $G = (V, E)$.



- a) Show that there is no Euler trail (i.e. a path contains each edge in E exactly once).
- b) Show that there are two paths, such that each edge in E appear in one of them exactly once.
- c) Find a minimal spanning tree of G .
- d) Show that there is only one minimal spanning tree of G .

Problem 7 [10' + 5']

There is a relation on $\mathbb{Z}$ that $x \sim y$ if and only if $2015|x - y$ (i.e. $x - y$ can be divided by 2015).

- a) Show that such a relation is an equivalence relation (i.e. reflexive, symmetric and transitive). In particular, how many equivalent classes are there in $\mathbb{Z}$ under this equivalence relation?
- b) Show that there exists an integer m such that $2015 \mid 11^m - 1$.

Hint: use Pigeonhole Principle for (b).

Problem 8 [5] Consider a binary tree T with n leaves. Define a function by

- $C(v) = 1$, for each leaf v , and
- $C(v)$ be the number of leaves that are descendants of v , for each internal vertex v .

Now, as each internal vertex v has two children w and u , let $X(v) = C(w) \cdot C(u)$. Show that the sum of $X(v)$, for all internal vertex v in T , equals $\binom{n}{2}$, i.e.

$$\sum_{\text{interval vertex } v} X(v) = \binom{n}{2}.$$