

Problem 1

Find all solutions to $(z^2 + iz + 6)(z - 1) = 0$, for $z \in \mathbb{C}$ and sketch these in the complex plane.

Problem 2

We are given the basis $\mathcal{B} = \{u, v, w\}$ for the vector space V and a linear transformation $T : V \rightarrow V$, given by $T(u) = 2u - 4w$, $T(v) = u + v + w$, $T(w) = -2u + \frac{1}{2}v$.

- a) What is the matrix representation of T with respect to the basis $\mathcal{B}$?

Hint: express u as $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}_{\mathcal{B}}^T$ and similarly with v and w .

- b) Find $T(u - 3w + 2v)$ expressed in coordinates with respect to the basis $\mathcal{B}$.
- c) What is the kernel and the image of T ? Determine whether T is surjective and/or injective.

Problem 3

- a) For which $a \in \mathbb{R}$ is the matrix $A = \begin{bmatrix} 1 & -3 & 0 \\ -1 & a & 2 \\ 2 & 0 & a - 2 \end{bmatrix}$ invertible?

- b) For which $a \in \mathbb{R}$ does the system of equations
$$\begin{cases} x - 3y & = -2 \\ -x + ay + 2z & = 0, \\ 2x & + (a - 2)z = -1 \end{cases}$$

have i) exactly one solution? ii) no solutions? iii) infinitely many solutions?

Problem 4

Let $T : \mathbb{C} \rightarrow \mathcal{M}_2$ be given by $T(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $z = a + bi$, where $\mathcal{M}_2$ is the vector space of real (2×2) -matrices.

- a) Show that T is a linear transformation.
- b) Show that T preserves multiplication (that is, $T(zw) = T(z)T(w)$, for $z, w \in \mathbb{C}$, where right-hand side indicates matrix multiplication).

- c) Show that $T(z)$ can be written as $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ and find $z \in \mathbb{C}$ such that $T(z) = AB$, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

Problem 5

- a) Show that $u = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ are linearly independent.
- b) What is the shortest distance between u and $V = \text{Sp}\{v, w\}$?
- c) Make a sketch of u, v, w, V and $\text{Proj}_V(u)$.

Problem 6

- a) Show that $\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$ is an inner product in $\mathcal{P}_2$, the vector space of polynomials of degree less than or equal to 2.
Hint: show linearity, symmetry and positive-definiteness.
- b) Find an orthogonal basis for the subspace U spanned by the vectors $\{1 - x, -1 + x^2\}$ in $\mathcal{P}_2$ with respect to the inner product in a).

Problem 7

Given the data points $(1, 4)$, $(2, 1)$ and $(-1, 0)$.

- a) Find a second order polynomial, in the form $p(x) = a_1x^2 + b_1x + c_1$, that passes through these points.
- b) Use the method of least squares to find the line $g(x) = a_2x + b_2$ that best fits the data.

Problem 8

Santa's gift-making machine is broken and will only produce three different gifts this year: Playstation, Barbie dolls and accordions. Moreover, it produces only one gift each day, but Santa has figured out a pattern:

- If one day the machine produces a PlayStation, there is a $1/4$ chance for it producing a new PlayStation the following day, $1/2$ chance for it producing a Barbie doll and $1/4$ chance for it producing an accordion.
 - The day after it produces a Barbie doll, the machine always produces an accordion.
 - It never produces accordions on two consecutive days, but is equally likely to produce a Barbie doll or a PlayStation the day after it produces an accordion.
- a) Find a stochastic matrix M representing how the Christmas present production changes from day to day and decide whether the matrix is regular.
- b) If the machine produces a PlayStation on a given day, what is the probability for there being produced a Barbie doll three days later?
- c) Luckily, it is a long time until Christmas, so there will be produced presents for all nice children. What will the distribution of Playstation, Barbie dolls and accordions be in the long run?

Problem 9

Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$.

- a) Is A diagonalizable? Explain your answer.
- b) Use the diagonalization of A to find A^5 .

Problem 10

Find the general solution of $y''(t) - 2y'(t) + y(t) = -3\cos(t)$ and solve for the initial conditions $y(0) = 2$ and $y'(0) = 0$.