



Department of Mathematical Sciences

Examination paper for **TMA4120 Calculus 4K**

Academic contact during examination:

Phone:

Examination date:

Examination time (from–to):

Permitted examination support material: (Code C): Approved simple calculator.

Other information:

- Every answer must be justified; describe clearly how you have reached your answers.

Language: English

Number of pages: 9

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave

Originalen er:

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Date

Signature

Problem 1 Find the solution $y(t)$ ($t > 0$) of the following initial value problem by using the Laplace transform:

$$y''(t) + 3y'(t) + 2y = f(t) = \begin{cases} 0, & 0 < t < 1, \\ 1, & 1 < t < 2, \\ 0, & 2 < t, \end{cases} \quad y(0) = y'(0) = 0.$$

Problem 2 Let $f(x) = x(1 - x)$ for $0 < x < 1$.

a) Find the Fourier cosine series with period 2 which represents $f(x)$ in $0 < x < 1$. Sketch the sum of the series on the interval $-2 < x < 2$.

b) Let $g(x) = \sum_{n=1}^{\infty} na_n \cos(n\pi x)$, and $h(x) = \sum_{n=1}^{\infty} n^2 a_n \cos(n\pi x)$, where

$$a_n = \frac{-2(1 + \cos(n\pi))}{(n\pi)^2}.$$

Check whether the following two integrals are finite:

$$(i) \int_{-1}^1 |g(x)|^2 dx, \quad (ii) \int_{-1}^1 |h(x)|^2 dx.$$

Hint: Parseval's identity.

c) Find a solution for the following heat equation with Neumann boundary conditions:

$$\begin{aligned} u_t(x, t) &= c^2 u_{xx}(x, t), & x &\in [0, 1], \quad t > 0, \\ u_x(0, t) &= u_x(1, t) = 0, & t &> 0, \\ u(x, 0) &= f(x) = x(1 - x), \end{aligned}$$

where c is a positive constant. Explain the details. *Hint:* Separation of variables.

Problem 3

- a) For every $z \in \mathbb{C}$ we define $f(z)$ by

$$f(z) = e^{-z^2}.$$

Use the Cauchy–Riemann equations to check if $f(z)$ is analytic in every point $z \in \mathbb{C}$.

- b) For every $z \in \mathbb{C}$ we define $g(z)$ by

$$g(z) = h(z) + \frac{1}{(z-1)^2},$$

where

$$h(z) = \begin{cases} e^{-(1/z^2)}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

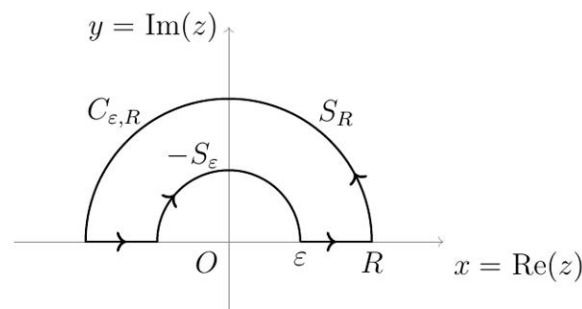
Find and classify all zeros and singularities of $g(z)$ including at infinity. Is $h(z)$ continuous at $z = 0$? Explain.

- c) Find all Taylor and Laurent series of $g(z)$ around $z = 0$. State the domains where the series converge.

Problem 4 Calculate the following integral

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx.$$

Hint: Integrate $f(z) = e^{iz}/z$ over the closed path $C_{\varepsilon,R}$,



where $S_r : re^{i\theta}$, $\theta \in [0, \pi]$ is a half circle of radius $r > 0$, oriented counter clockwise.

You may use without explanation Jordan's Lemma: If $f(z) = e^{iz}g(z)$ for a continuous function $g(z)$ on S_R , then

$$\left| \int_{S_R} f(z) dz \right| \leq \pi \max_{z \in S_R} |g(z)|.$$