



Department of Mathematical Sciences

Examination paper for **TMA4130 Mathematics 4N**

Academic contact during examination:

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Examination date: December 6, 2023

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- Approved calculator
- One yellow, stamped A5 sheet of self-written notes

Language: English

Number of pages: 15

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave**Originalen er:****1-sidig** ☐ **2-sidig** ☒**sort/hvit** ☒ **farger** ☐**skal ha flervalgskjema** ☐

Date

Signature

Problem 1 Interpolation [10 pts]Find the trigonometric polynomial $t(x)$ of the form

$$t(x) = c_0 + c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x,$$

which interpolates the following values

$$\begin{array}{c|cccccc} x_i & -\pi/2 & -\pi/4 & 0 & \pi/4 & \pi/2 \\ \hline y_i & 2 & \sqrt{2} & 0 & \sqrt{2} & 2 \end{array}.$$

Problem 2 Numerical integration [10 pts]

Although the 2π -periodic function $f(x) = 2^{\sin x}$ can be written as a Fourier series, most of its coefficients a_n, b_n can only be computed numerically. Using the composite Simpson's rule with two subintervals ($[-\pi, 0]$ and $[0, \pi]$), approximate a_0 and b_1 .

Problem 3 Numerics for Nonlinear Equations [10 pts]

Consider the initial value problem

$$y'(t) = \cos(2ty), \quad y(0) = 0.$$

Using a certain Runge–Kutta method with time-step size $h = 0.5$, the numerical approximation of $y(t)$ at time $t_{n+1} = (n+1)h$ will be

$$y_{n+1} = y_n + 0.5 \cos[(n+1)y_{n+1}].$$

- a) Which Runge–Kutta method is being used above?
- b) To compute the first time step, we have to solve a nonlinear equation:

$$y_1 = 0.5 \cos y_1.$$

Renaming y_1 as x , the nonlinear equation can be written as

$$2x - \cos x = 0. \tag{6}$$

Using Newton's method to solve Eq. (6) numerically, with initial guess $x_0 = 0$, iterate until the absolute difference between two consecutive iterations is less than 0.001 (in other words, iterate until $|x_{k+1} - x_k| < 10^{-3}$).

Problem 4 Laplace transform [10 pts]

- a) Solve the following ordinary differential equation using the Laplace transform:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y(t) = 2e^{2t}.$$

Show all the steps in your solution, including the application of the Laplace transform and shifting theorems (if needed), and clearly state the inverse transform to obtain the final solution for $y(t)$. Note that your solution $y(t)$ should contain the (at this point unknown) initial conditions.

- b) Given the initial conditions $y(0) = 1$, $y'(0) = 3$, find the solution to the differential equation you solved in part a).

Problem 5 Numerical methods for ODEs [10 pts]

```
1 import numpy as np
2
3 def F(t,y):
4     return np.array([ y[1], -y[0] ])
5
6 t0, tend = 0.0, 20.0
7 y0 = np.array([1,1])
8 h=0.01
9 N = int(np.round((tend-t0)/h))
10 h = (tend-t0)/N
11 Y = np.zeros((2,N+1))
12 Y[:,0] = y0
13 T = np.linspace(t0, tend, N+1)
14
15 for k in range(N):
16     K1 = F(T[k],Y[:,k])
17     Y[:,k+1] = Y[:,k] + h/2*(K1 + F(T[k+1], Y[:,k] + h*K1))
```

- a) The code above was written for approximating the solution to some initial value problem. Determine which initial value problem it is, and the method which is used.
- b) A second order method was used to solve the problem in the code above. The step size used was $h = 0.01$ and the norm of the error at $t = 20$ turned out to be $\approx 0.47 \cdot 10^{-3}$. Give an estimate for how big the norm of the error would have been at $t = 20$ if we had chosen $h = 0.05$ instead.

Problem 6 Fourier series [10 pts]

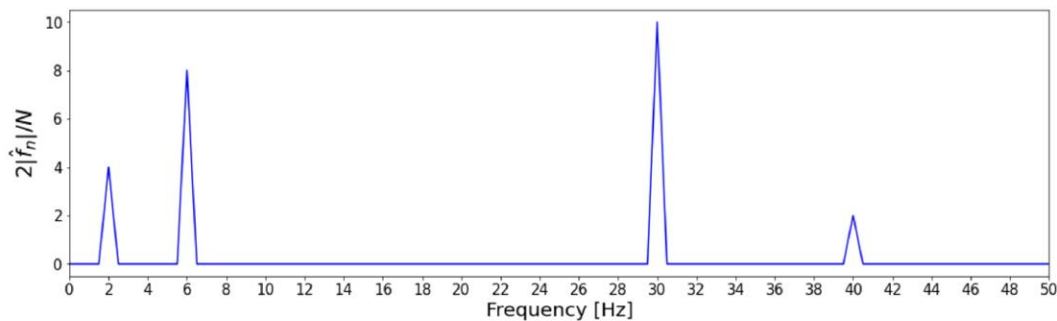
Calculate the Fourier series of the function $f(x) = |x^3|$ defined on $[-\pi, \pi]$. Explicitly write down the first five non-vanishing terms of the Fourier sum. **Hint:** Note that $|x^3|$ is an even function. The formulas below, where $\alpha \in \mathbb{R}$, may also be useful:

$$\int x^3 \sin(\alpha x) dx = \frac{3(\alpha^2 x^2 - 2) \sin(\alpha x) - \alpha x(\alpha^2 x^2 - 6) \cos(\alpha x)}{\alpha^4} + \text{constant}$$

$$\int x^3 \cos(\alpha x) dx = \frac{\alpha x(\alpha^2 x^2 - 6) \sin(\alpha x) + 3(\alpha^2 x^2 - 2) \cos(\alpha x)}{\alpha^4} + \text{constant}$$

Problem 7 Discrete Fourier Transform, 4D,[10 pts]

The spectrum of a certain signal f , sampled at regular time intervals, is shown below (as usual, the “mirrored” half of the spectrum is omitted from the plot):



The following Python code computes that spectrum using the `fft` function, then applies a certain type of filter to the original signal:

```
1 import numpy as np
2
3 N = f.size #(the signal f was pre-computed elsewhere)
4 dt = 0.001 #spacing between two samples, in seconds
5 L = N*dt #length of the sampling interval, in seconds
6
7 n = np.arange(N) #vector of indices n = 0,1,...,N-1
8 nShifted = N/2-np.abs(n-N/2) #vector of shifted indices
9 shiftedFreq=nShifted/L #shift frequencies to account for mirroring
10
11 cutoffFreq = 25 #in Hertz
12 DFT = np.fft.fft(f)
13 DFT[shiftedFreq < cutoffFreq] = 0
14 g = np.fft.ifft(DFT)
```

- a) What kind of filter is implemented above?
- b) Among the functions below, which one could best represent the *filtered* signal?

- i) $g(t) = 4 \cos(4\pi t) + 8 \sin(12\pi t)$
- ii) $g(t) = \cos(4\pi t) + \sin(12\pi t)$
- iii) $g(t) = \cos(60\pi t) + \sin(80\pi t)$
- iv) $g(t) = 10 \cos(60\pi t) + 2 \sin(80\pi t)$

Remember: if t is measured in seconds, and $\alpha > 0$, then the frequency (in Hertz) of a function such as $\cos(2\pi\alpha t)$ or $\sin(2\pi\alpha t)$ is simply α .

Problem 7 Fourier transform [10 pts], 4N

Let

$$g(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}.$$

- a) Compute the Fourier transform of g .
- b) Compute the Fourier transform of $f = g * g$.

Problem 8 Wave Equation [10 pts]

Use the d'Alembert formula to find the solution to the wave equation $u_{tt} = 9u_{xx}$ with initial conditions $u(0, x) = \sin x$, $u_t(0, x) = \cos x$

Problem 9 Heat Equation [10 pts]

For the one-dimensional heat equation $u_t = u_{xx}$ on $[0, \infty) \times [0, 2]$, use the method of separation of variables to find the solution satisfying the Neumann boundary conditions $u_x(t, 0) = u_x(t, 2) = 0$ and the initial condition $u(0, x) = 1 - \cos 3\pi x$.