

Department of Mathematical Sciences

Problem 1 Laplace transform [10 pts]

- a) Compute the Laplace transform of

$$f(t) = 3 \sin(2t)e^t.$$

- b) Compute the inverse Laplace transform of

$$F(s) = \frac{3s^2 - 11s + 12}{s(s-2)^2}.$$

- c) Use the Laplace transform to solve the initial value problem

$$y'(t) + 3y(t) = \exp(2t), \quad y(0) = 3.$$

Problem 2 Fourier series [10 pts] Extend the function

$$f(x) = 2x,$$

defined on the interval $[0, 2)$, to the interval $[0, 4)$ in such a way that the periodic extension of this function is **even** with a period of 4. Find the Fourier series of this new function, and explicitly write down the first 4 nonvanishing terms of the Fourier series.

Problem 3 Numerical methods for ODEs [10 pts] Consider python code that solves an initial value problem with a Runge-Kutta method and prints the resulting x and y value after N steps:

```
1 def f(x, y):
2     return x * y**2
3
4 def step(x, y, h):
5     k1 = f(x, y)
6     k2 = f(x + h, y + h * k1)
7     y_new = y + (h / 2) * (k1 + k2)
8     x_new = x + h
9     return x_new, y_new
10
11 x = 0
12 y = 0.5
13 h = 0.1
14 N=20
15
16 for n in range(N):
17     x,y=step(x,y,h)
18
19 print(x,y)
```

- a) What is the initial value problem that the code solves?
- b) Write the Butcher tableau for the method used. Is the method explicit or implicit?

motivate your answer.

- c) Compute one step of the algorithm in the code by hand. That is, what is the output if we run the code with $N = 1$ instead?

Problem 4 Interpolation [10 pts] Consider the data points

$$\begin{array}{c|ccc} x_i & 0 & 2 & 4 \\ \hline y_i = f(x_i) & 1 & 0 & 1 \end{array}$$

- a) Use Lagrange interpolation to find the polynomial of minimal degree interpolating these points. Express the polynomial in the form

$$p_n(x) = a_n x^n + \cdots + a_1 x + a_0.$$

- b) Determine the Newton form of the interpolating polynomial.
c) Verify that the solutions in (b) and (c) are the same.

I suppose that this should be "Verify that the solutions in (a) and (b) are the same."

- d) Use your result to find an approximation to $f(1)$.

Problem 5 Heat Equation [10 pts] Solve the problem

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial t} u(t, x) = 9 \frac{\partial^2}{\partial x^2} u(t, x), & \text{for } 0 \leq t, 0 \leq x \leq \pi, \\ \frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(\pi, 0) = 0 & \\ \text{correction: } = \frac{\partial}{\partial x} u(t, \pi) = 0, & \text{for } 0 \leq t, \\ u(0, x) = 3 \cos(x) - \cos(3x), & \text{for } 0 \leq x \leq \pi, \end{array} \right.$$

using separation of variables.

Problem 6 Wave Equation [10 pts]

- a) Verify by explicit computations that the function:

$$\hat{u}(x, t) = \sin\left(\frac{m\pi}{L}x\right) \cos\left(c\frac{m\pi}{L}t\right)$$

and the function

$$\tilde{u}(x, t) = \frac{1}{2} \left(\sin\left(\frac{m\pi}{L}(x - ct)\right) + \sin\left(\frac{m\pi}{L}(x + ct)\right) \right)$$

are both solutions of the 1-D wave equation

$$u_{tt} = c^2 u_{xx},$$

with initial conditions conditions (initial shape) $u(x, 0) = f(x) = \sin\left(\frac{m\pi}{L}x\right)$ and (vibration from rest) $u_t(x, 0) = g(x) = 0$.

- b) Show that $\hat{u} = \tilde{u}$.

Hint: Use a trigonometric identity.

- c) Suppose that the functions f and g have the following symmetry properties for $0 \leq x \leq L$: $f(L - x) = f(x)$ and $g(L - x) = g(x)$, i.e., both are symmetric around $2x = L$. Use d'Alembert's solution of the 1-D wave equation

$$u(x, t) = \frac{1}{2} \left(f^*(x + ct) + f^*(x - ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g^*(s) ds \right),$$

where f^* and g^* are the ($2L$ -periodic) odd extensions of f respectively g , to show for all $0 < x < L$ and $t > 0$ that

$$u(x, t + \frac{L}{c}) = -u(x, t).$$

Problem 7 Fourier Transform [10 pts]

Suppose the function f is integrable and even. Consider the function $g(t) = \cos(at)$. Show that for all $a \in \mathbb{R}$, we have that the convolution product

$$(f * g)(t) = \cos(at) \hat{f}(a).$$

Recall that $\hat{f}(a) = \mathcal{F}(f)(a)$ is the Fourier transform of f evaluated at a .

Hint: You may use the trigonometric identity $\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$.

For Mathematics 4D: there should be an extra factor $\sqrt{2\pi}$ in the answer (i.e. they should show $(f * g)(t) = \sqrt{2\pi} \cos(at) \hat{f}(a)$) because the convolution was defined without $\frac{1}{\sqrt{2\pi}}$ in the lectures.

Problem 8 Numerical integration [10 pts]

- a) Find the values of w_j , $j = -1, 0, 1, 2$, such that the quadrature rule

$$Q[f] = w_{-1}f(-1) + w_0f(0) + w_1f(1) + w_2f(2)$$

gives the correct value for the integral

$$\int_0^1 f(s) ds,$$

when f is any polynomial of degree three.

- b) Denote by T_1 the trapezoid rule using a single subinterval. Denote by M_1 the midpoint rule using a single subinterval. Denote By S_2 the Simpson's rule using a single subinterval. Show that, for any continuous function f , and any interval $[a, b]$

$$S_2[f](a, b) = \frac{1}{3} T_1[f](a, b) + \frac{2}{3} M_1[f](a, b).$$

- c) c1) Show that the quadratic Legendre polynomial $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is orthogonal over $[-1, 1]$ to all linear polynomials.
c2) The first two Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x.$$

It can be shown that the other Legendre polynomials, P_{n+1} , $n > 1$, satisfy the recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

Use this to show by induction that the leading coefficient for the Legendre polynomials is

$$k_n = \frac{(2n)!}{2^n (n!)^2}$$

Problem 9 Numerics for Nonlinear Equations [10 pts] Let r be a solution (root) of the following equation

$$x - \cos(3x) = 0, \quad 0 \leq x \leq 1.$$

- a) Show that a root exists by using the intermediate value theorem, and show that the solution is unique.
- b) Perform two iterations of the bisection method on the interval $[0, 1]$. What is an upper bound of the error after these 2 iterations?
- c) How many iterations are necessary in order to compute the root up to the tolerance $tol = 0.01$?