Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4130/35 Mathematics 4N/4D**

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Examination date: December 02 2019

Examination time (from–to): 09:00–13:00

Permitted examination support material: Code C: Approved calculator
One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides)

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- There are two versions of Problem 3: one for Mathematics 4N and one for Mathematics 4D.
- Good Luck!

Language: English

Number of pages: 7

Number of pages enclosed: 2

Checked by:

Informasjon om trykking av eksamensoppgave	
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Problem 1 [20 points]

a) Compute Laplace transform of

$$f(t) = \begin{cases} 0 & 0 \le t \le 1, \\ t & t > 1. \end{cases}$$

We have

$$F(s) = \int_1^\infty te^{-st}\, dt = t\frac{e^{-st}}{-s}\Big|_1^\infty - \int_1^\infty \frac{e^{-st}}{-s}\, dt, \quad s > 0$$

Hence

$$F(s) = e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right).$$

b) Use Laplace transform to find the solution of

$$y'' + y = 2e^t, \qquad \text{with } y(0) = y'(0) = 0.$$

Apply the Laplace transform, we get

$$s^2Y + Y = \frac{2}{s-1}.$$

Thus partial fraction decomposition gives

$$Y = \frac{2}{(s-1)(s^2+1)} = \frac{1}{s-1} - \frac{s+1}{s^2+1}.$$

Hence

$$y = e^t - \cos t - \sin t.$$

c) Compute the inverse Laplace transform $\mathcal{L}^{-1}(F)(t)$ of the following function

$$F(s) = \frac{1}{s^2 + 2s + 17}.$$

Since

$$F(s) = \frac{1}{(s+1)^2 + 4^2},$$

we have

$$f(t) = e^{-t}\frac{\sin 4t}{4}.$$

Problem 2 [14 points]

a) Let

$$f(x) = 1 + x, \quad -\pi < x < \pi.$$

Verify the following complex Fourier series expansion for f

$$1 + \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx}.$$

We have that $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + x)\, dx = 1$. Thus it is enough to show, for $n \neq 0$, we have $c_n = \frac{i(-1)^n}{n}$. In fact

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx = \frac{x e^{-inx}}{-2\pi i n}\Big|_{-\pi}^{\pi} = \frac{i(-1)^n}{n}.$$

b) Why is

$$f(x) = 1 + \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx}$$

for $-\pi < x < \pi$?

Because f is smooth, that is, f is continuously differentiable on the interval $-\pi < x < \pi$.

c) Compute the Fourier transform of

$$f(x) = \begin{cases} \sin(x) & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

We have

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\omega} \sin x\, dx.$$

Notice that

$$e^{-ix\omega} = \cos x\omega - i \sin x\omega,$$

$\cos x\omega \sin x$ is odd and $\sin x\omega \sin x$ is even. Thus

$$\int_{-1}^{1} e^{-ix\omega} \sin x\, dx = 2(-i) \int_{0}^{1} \sin x\omega \sin x\, dx.$$

Recall that

$$2\sin a \sin b = \cos(a-b) - \cos(a+b).$$

Thus

$$\int_{-1}^{1} e^{-ix\omega} \sin x \, dx = (-i) \int_{0}^{1} \cos(x\omega - x) - \cos(x\omega + x) \, dx,$$

which gives

$$\int_{-1}^{1} e^{-ix\omega} \sin x \, dx = (-i) \left(\frac{\sin(x\omega - x)}{\omega - 1}\Big|_0^1 - \frac{\sin(x\omega + x)}{\omega + 1}\Big|_0^1 \right).$$

Thus

$$\hat{f}(\omega) = \frac{-i}{\sqrt{2\pi}} \left(\frac{\sin(\omega - 1)}{\omega - 1} - \frac{\sin(\omega + 1)}{\omega + 1} \right).$$

Problem 3 *TMA4130 Mathematics 4N:* [6 points]

Solve the initial value problem for the wave equation ($u_{tt} = \partial^2 u/\partial t^2$, $u_{xx} = \partial^2 u/\partial x^2$ mean partial derivatives)

$$u_{tt} = u_{xx}, \quad u(x,0) = \sin(x), \quad u_t(x,0) = e^x,$$

using d'Alembert's solution.

We have

$$u(x,t) = \frac{1}{2}(\sin(x+t) + \sin(x-t) + e^{(x+t)} - e^{(x-t)}).$$

Problem 3 *TMA4135 Mathematics 4D:* [6 points]

Show that the following function $u(x,t) = (x-t)^3 + \sin(x+t)$ satisfies the wave equation $u_{xx} = u_{tt}$ ($u_{tt} = \partial^2 u/\partial t^2$, $u_{xx} = \partial^2 u/\partial x^2$ mean partial derivatives).

A direct computation gives

$$u_{xx} = 6(x-t) - \sin(x+t) = u_{tt}.$$

Problem 4 [20 points]

Consider the following heat equation

$$u_t(x,t) = \frac{1}{2}u_{xx}(x,t), \quad t \geq 0, \quad 0 \leq x \leq \pi,$$

($u_t = \partial u/\partial t,\ u_{xx} = \partial^2 u/\partial x^2$ mean partial derivatives) with boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0;$$

and the initial condition

$$u(x,0) = x(\pi - x), \quad 0 \leq x \leq \pi.$$

a) Find the Fourier sine series solution of the above heat equation by using the separation of variables method.

Separating variable method gives a Fourier sine series solution

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}n^2 t} A_n \sin nx,$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx.$$

Integration by parts gives

$$A_n = \frac{2}{\pi n}\int_0^{\pi} (\pi - 2x)\cos nx\, dx = \frac{4}{\pi n^2}\int_0^{\pi} \sin nx\, dx = \frac{4(1-(-1)^n)}{\pi n^3}.$$

b) Let M, N be two natural numbers, and define $h = \pi/M$ and $k = 1/N$. Introduce $x_i = ih$ for $i = 0, \ldots, M$ and $t_n = nk$ for $n = 0, 1, 2, \ldots$. Write down an explicit difference scheme (based on finite differences and (forward) Euler's method) for $U_i^n \approx u(x_i, t_n)$.

Solution: Symmetric approximation to u_{xx} and forward Euler for u_t yields

$$U_i^{n+1} = U_i^n + \frac{k}{2h^2}\Big(U_{i+1}^n - 2U_i^n + U_{i-1}^n\Big), \quad i = 1, \ldots, M-1,\ n = 0, 1, 2, \ldots,$$

with boundary conditions

$$U_0^n = 0, \quad U_M^n = 0, \quad n = 0, 1, 2, \ldots,$$

and initial conditions

$$U_i^0 = x_i(\pi - x_i) \quad i = 0, \ldots, M.$$

c) Let $M = 4$ and $N = 20$, and compute approximate solutions for $u(\pi/4, 0.1)$

Compute U_1^2, the above scheme gives

$$u(\pi/4, 0.1) \approx U_1^2 = U_1^1 + \frac{2}{5\pi^2}(U_2^1 - 2U_1^1),$$

where

$$U_1^1 = \frac{3\pi^2}{16} + \frac{2}{5\pi^2}(-\frac{\pi^2}{8}) = \frac{3\pi^2}{16} - \frac{1}{20}$$

and

$$U_2^1 = \frac{\pi^2}{4} + \frac{2}{5\pi^2}(-\frac{\pi^2}{8}) = \frac{\pi^2}{4} - \frac{1}{20}.$$

Thus

$$u(\pi/4, 0.1) \approx \frac{3\pi^2}{16} - \frac{1}{10} + \frac{1}{50\pi^2} \approx 1.7525772488.$$

Problem 5 [10 points]

Find a, b, c, d such that the polynomial

$$p(x) = ax^3 + bx^2 + cx + d$$

interpolating the points

x_i	0	2	3	4
y_i	1	5	10	17

Use the Lagrange method or the Newton method. The result is $p(x) = x^2 + 1$.

Problem 6 [10 points]

The integral

$$\int_0^1 f(x)\,dx,$$

can be approximated by the Simpson formula

$$S = \frac{1}{6}\Big(f(0) + 4f(0.5) + f(1)\Big).$$

a) Apply the Simpson formula to the integral

$$\int_0^1 x^3\,dx.$$

$$S = \frac{1}{6}\Big(0 + 0.5 + 1\Big) = 0.25 = \int_0^1 x^3\,dx.$$

b) Determine the degree of precision for the Simpson formula.?

3, because

$$S(x^4) = \frac{1}{6}\Big(0 + 0.25 + 1\Big) = \frac{5}{24} \neq \int_0^1 x^4\,dx = \frac{1}{5}.$$

Problem 7 [10 points]

Let r be the solution of the following equation

$$x + \ln(x-1) = 0, \quad 1 < x < 2.$$

Show that the solution is unique. Starting from

$$x_0 = 1.25,$$

do one time Newton iteration, and compute x_1.

Put

$$f(x) = x + \ln(x-1),$$

then

$$f'(x) = 1 + \frac{1}{x-1} > 1$$

when $x > 1$. But $f(1+) = -\infty < 0$, $f(2) = 2 > 0$. Thus we have exact one solution inside $(1, 2)$. By Newton's method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{5 + \ln 4}{5} \approx 1.27725887222.$$

Problem 8 [10 points]

The following Heun's method is given:

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(x_n, \mathbf{y}_n), \\
\mathbf{k}_2 &= \mathbf{f}(x_n + h, \mathbf{y}_n + h\mathbf{k}_1), \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{h}{2}(\mathbf{k}_1 + \mathbf{k}_2).
\end{aligned}$$

a) Do one step with step size $h = 0.1$ using the above method on the problem:

$$y' = -2xy, \quad y(0) = 1.$$

Find the exact solution of the above equation and compute the error.

$k_1 = 0$, $k_2 = -0.2$, thus $y_1 = 0.99$. The exact solution is

$$y = e^{-x^2}.$$

The error is

$$e^{-0.01} - 0.99$$

b) Find the stability function $R(z)$ for Heun's method. Find also the corresponding stability interval.

$R(z) = 1 + z + z^2/2, \quad [-2, 0]$

Fourier Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} \, \mathrm{d}w$	$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, \mathrm{d}x$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{w^2 + a^2}$
$\frac{1}{x^2 + a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert w\rvert}}{a}$
$\begin{cases} 1 & \text{for } \lvert x\rvert < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin wa}{w}$

Laplace Transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t) \, \mathrm{d}t$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, for $n = 0, 1, 2, \ldots$, $\Gamma(n+1) = n!$
e^{at}	$\frac{1}{s - a}$
$\delta(t - a)$	e^{-as}

$$\int x^n \cos ax \, \mathrm{d}x = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, \mathrm{d}x$$

$$\int x^n \sin ax \, \mathrm{d}x = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, \mathrm{d}x$$

Numerics

- Newton's method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.
- Newton's method for system of equations: $\vec{x}_{k+1} = \vec{x}_k - JF(\vec{x}_k)^{-1}F(\vec{x}_k)$, with $JF = (\partial_j f_i)$.
- Lagrange interpolation: $p_n(x) = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$, with $l_k(x) = \prod_{j\neq k}(x - x_j)$.
- Interpolation error: $\epsilon_n(x) = \prod_{k=0}^{n}(x - x_k)\frac{f^{(n+1)}(t)}{(n+1)!}$.
- Chebyshev points: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \leq k \leq n$.
- Newton's divided difference: $f(x) \approx f_0 + (x - x_0)f[x_0, x_1] +$ $(x-x_0)(x-x_1)f[x_0, x_1, x_2] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_0, \ldots, x_n]$, with $f[x_0, \ldots, x_k] = \frac{f[x_1,\ldots x_k] - f[x_0,\ldots,x_{k-1}]}{x_k - x_0}$.
- Trapezoid rule: $\int_a^b f(x)\,\mathrm{d}x \approx h\left[\frac{1}{2}f(a) + f_1 + f_2 + \cdots + f_{n-1} + \frac{1}{2}f(b)\right]$.
  Error of the trapezoid rule: $|\epsilon| \leq \frac{b-a}{12}h^2 \max_{x\in[a,b]}|f''(x)|$.
- Simpson rule: $\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n\right]$.
  Error of the Simpson rule: $|\epsilon| \leq \frac{b-a}{180}h^4 \max_{x\in[a,b]}|f^{(4)}(x)|$.
- Gauss–Seidel iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(m+1)} - \mathbf{U}\mathbf{x}^{(m)}$, with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(m)}$.
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Improved Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}^*_{n+1})]$, where $\mathbf{y}^*_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Classical Runge–Kutta method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$,
  $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$, $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$,
  $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$, $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$.
- Finite differences: $\frac{\partial u}{\partial x}(x, y) \approx \frac{u(x+h,y) - u(x-h,y)}{2h}$, $\frac{\partial^2 u}{\partial x^2}(x, y) \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$.
- Crank–Nicolson method for the heat equation: $r = \frac{k}{h^2}$,
  $(2 + 2r)u_{i,j+1} - r(u_{i+1,j+1} + u_{i-1,j+1}) = (2 - 2r)u_{ij} + r(u_{i+1,j} + u_{i-1,j})$.