English

UNIVERSITY OF BERGEN

The Faculty of Mathematics and Natural Sciences

Solutions to Exam in MAT111 - Calculus 1

Wednesday May 10, 2017, 09.00-14.00

Exercise 1.

a) Find the real numbers x and y such that the complex variable $z = x + iy$ solves the system

$$\begin{cases} |z+i| = |z-1|, \\ |z| = 4. \end{cases}$$

Draw on the complex plane the set of the points that satisfies to each of the equation in the system. Show on the complex plane the solution of the system.

b) Write the complex number $(-1-i)^6$ in the form $re^{i\theta} = r\cos\big(\cos\theta + i\sin\theta\big)$ (polar form) and in the form $a + ib$ (cartesian form).

c) Find all the complex numbers w satisfying the equation $w^4 = -16$ in both polar and cartesian forms and show the answer on the complex plain.

Solution to a). We substitute $z = x + iy$ into the first equation and obtain

$$\begin{aligned} |z+i|^2 &= |z-1|^2 \\ |x+i(y+1)|^2 &= |(x-1)+iy|^2 \\ x^2+y^2+2y+1 &= x^2-2x+1+y^2 \\ y &= -x. \end{aligned}$$

Substituting $z = x + iy$ into the second equation, we obtain

$$|z|^2 = 16 \quad \Longrightarrow \quad x^2 + y^2 = 16.$$

Thus we need to solve the system

$$\begin{cases} y = -x, \\ x^2 + y^2 = 16. \end{cases}$$

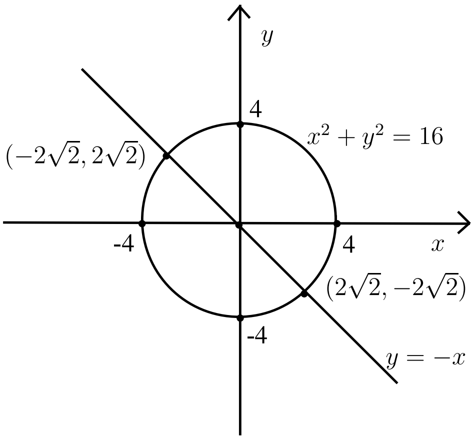
We deduce

$$\begin{cases} y = -x, \\ x^2 + y^2 = 16, \end{cases} \implies \begin{cases} y = -x, \\ 2x^2 = 16, \end{cases} \implies \begin{cases} y = -x, \\ x = \pm 2\sqrt{2}. \end{cases}$$

So we obtain two solutions

$$x = 2\sqrt{2},\ y = -2\sqrt{2}, \quad \text{and} \quad x = -2\sqrt{2},\ y = 2\sqrt{2}.$$

The set of points $|z+i| = |z-1|$ is the straight line $y = -x$ and $|z| = 4$ is the circle of radius 2 centered at the origin.



Solution to b). We write $z = -1-i$ in the polar form as $z = \sqrt{2}e^{i\frac{5\pi}{4}}$ and obtain

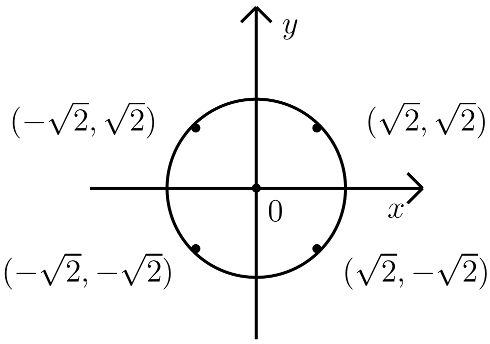
$$(-1-i)^6 = (\sqrt{2})^6 e^{i\frac{5\pi}{4}6} = 8e^{i\frac{3\pi}{2}} = 8\Big(\cos(\frac{3\pi}{2}) + \sin(\frac{3\pi}{2})\Big) = -8i.$$

Solution to c). We apply the formula of n-th root of $z = r(\cos\theta + i\sin\theta)$ and obtain:

$$\sqrt[4]{z} = \sqrt[4]{r}\Big(\cos\big(\frac{\theta + 2\pi k}{4}\big) + i\sin\big(\frac{\theta + 2\pi k}{4}\big)\Big).$$

Thus

$$\begin{aligned} w_0 &= 2\Big(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\Big) = \sqrt{2} + i\sqrt{2}, \\ w_1 &= 2\Big(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\Big) = -\sqrt{2} + i\sqrt{2}, \\ w_2 &= 2\Big(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})\Big) = -\sqrt{2} - i\sqrt{2}, \\ w_3 &= 2\Big(\cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4})\Big) = \sqrt{2} - i\sqrt{2}. \end{aligned}$$



Exercise 2.

a) Show that $\lim_{x\to 2+} \pi^{(x-2)(\ln(x-2))^2} = 1$.

b) Find the limit $\lim_{x\to i} \frac{x^2+2ix+3}{x^2+1}$ or show that the limit does not exists.

Solution to a). Since the exponential function is continuos, we need to show that

$$\pi^{\lim_{x\to 2+}(x-2)(\ln(x-2))^2} = 1 \quad \text{or} \quad \lim_{x\to 2+}(x-2)(\ln(x-2))^2 = 0.$$

It is equivalent to show $\lim_{x\to 0+} x(\ln x)^2 = 0$. We rewrite the latter expression and use the L'Hopital rule

$$\lim_{x\to 0+} x(\ln x)^2 = \lim_{x\to 0+} \frac{(\ln x)^2}{\frac{1}{x}} = \lim_{x\to 0+} \frac{\frac{2}{x}\ln x}{-\frac{1}{x^2}} = \lim_{x\to 0+} \frac{2\ln x}{-\frac{1}{x}}.$$

We use the L'Hopital rule again and obtain

$$\lim_{x\to 0+} \frac{2\ln x}{-\frac{1}{x}} = \lim_{x\to 0+} \frac{2\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to 0+} 2x = 0.$$

Solution to b). We calculate

$$\lim_{x\to i} \frac{x^2+2ix+3}{x^2+1} = \lim_{x\to i} \frac{(x-i)(x+3i)}{(x-i)(x+i)} = \lim_{x\to i} \frac{x+3i}{x+i} = 2.$$

Exercise 3.

a) Find the values of α and β such that the function

$$f(x) = \begin{cases} \alpha \cos x + 2, & \text{for} \quad x < 0, \\ \beta e^{3x} + \alpha x^2, & \text{for} \quad x \geq 0 \end{cases}$$

is continuos on the interval $(-\infty, +\infty)$.

b) For which values of α and β is the function differentiable?

Solution to a). We have $f(0) = \beta$. We calculate the limit of $f(x)$ as $x \to 0$ from the left and from the right:

$$\lim_{x \to 0-} \alpha \cos x + 2 = \alpha + 2 \quad \text{and} \quad \lim_{x \to 0+} \beta e^{3x} + \alpha x^2 = \beta.$$

We conclude that they are equal and coincide with $f(0) = \beta$ for all $\alpha + 2 = \beta$

Solution to b). The right derivative of $f(x)$ is

$$\lim_{x \to 0+} \frac{\beta e^{3x} + \alpha x^2 - \beta}{x} = \beta \lim_{x \to 0+} \frac{e^{3x} - 1}{x} = \beta \lim_{x \to 0+} \frac{3e^{3x}}{1} = 3\beta.$$

The left derivative is

$$\lim_{x \to 0-} \frac{\alpha \cos x + 2 - (\alpha + 2)}{x} = \alpha \lim_{x \to 0-} \frac{\cos x - 1}{x} = \alpha \lim_{x \to 0-} \frac{-\sin x}{1} = 0.$$

The right and the left derivatives coincide only for $\beta = 0$. Since the differentiable function have to be continuos we conclude that $\alpha = -2$.

Exercise 4.

Find the indefinite integrals

$$a) \int \tan(x) \ln(\cos x)\, dx, \qquad b) \int (x^2 - 2x + 5) e^{2x}\, dx.$$

Solution to a). We use the substitution $u = \cos x$ and $du = -\sin x$. Then

$$\int \tan(x) \ln(\cos x)\, dx = -\int \frac{\ln(\cos x)}{\cos x} (-\sin x)\, dx = -\int \frac{\ln u}{u}\, du.$$

Now we use the substitution $v = \ln u$ and $dv = \frac{du}{u}$ and obtain

$$-\int \frac{\ln u}{u}\, du = -\int v\, dv = -\frac{1}{2} v^2 = -\frac{1}{2} (\ln u)^2 = -\frac{1}{2} \big(\ln(\cos x) \big)^2.$$

The indefinite integral is equal to $-\frac{1}{2}\big(\ln(\cos x)\big)^2 + C$.

Solution to b). We calculate

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x}.$$

To calculate the integral $\int xe^{2x}\,dx$ we use the method integration-by-parts. We write $u = x$, $du = dx$ and $dv = e^{2x}$, $v = \frac{e^{2x}}{2}$ and obtain

$$\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}.$$

To calculate the integral $\int x^2e^{2x}\,dx$ we use the method integration-by-parts and reduce to the previous integral. First we write $u = x^2$, $du = 2xdx$ and $dv = e^{2x}$, $v = \frac{e^{2x}}{2}$. Then

$$\int x^2e^{2x}\,dx = \frac{x^2e^{2x}}{2} - \int xe^{2x}\,dx = \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4}.$$

Summing all the integrals, we obtain

$$\int (x^2 - 2x + 5)e^{2x}\,dx = \frac{x^2e^{2x}}{2} - 3\Big(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\Big) + \frac{5}{2}e^{2x} + C.$$

Exercise 5.

Consider the function $f(x) = e^{-x}(x^2 + 5x + 1)$ defined on $(-\infty, +\infty)$.

a) Find maximum and minimum of the function.

b) Find the asymptotes for the function.

c) Find the inflection points and indicate the intervals, where the function is concave and where the function is convex.

d) Draw the graph of the function.

Solution to a). We calculate $f'(x) = -e^{-x}(x^2+3x-4)$. The critical points are given by the zeros of the derivative and equal to $x = -4$ and $x = 1$. Since $f'(x) = -e^{-x}(x+4)(x-1)$ we conclude that

f decreases on $(-\infty, -4) \cup (1, +\infty)$,

f increases on $(-4, 1)$, and

$x = -4$ is the minimal point and $x = 1$ is the maximal point of $f(x)$.

Solution to b). There is no vertical asymptotes, since the domain of the definition of the function is $(-\infty, +\infty)$. To find other asymptotes we calculate

$$\lim_{x\to+\infty}\frac{e^{-x}(x^2+5x+1)}{x}=0 \quad\text{and}\quad \lim_{x\to-\infty}\frac{e^{-x}(x^2+5x+1)}{x}=-\infty.$$

We conclude that it could be a horizontal asymptote for $x\to+\infty$. We calculate

$$\lim_{x\to+\infty}e^{-x}(x^2+5x+1)=0,$$

and deduce that there is the horizontal asymptote $y=0$.

Solution to c). We calculate

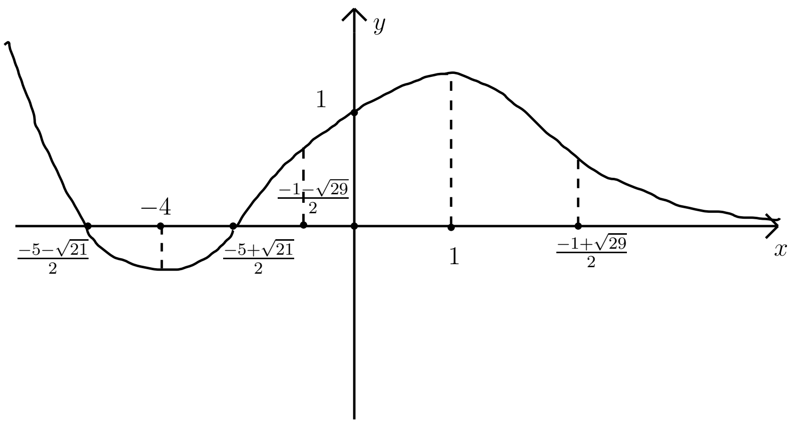
$$f''(x)=e^{-x}(x^2+x-7)=e^{-x}(x-\frac{-1-\sqrt{29}}{2})(x-\frac{-1+29}{2}).$$

The inflection points are $x=\frac{-1\pm\sqrt{29}}{2}$ and

f is concave below (convex) on $(-\infty,\frac{-1-\sqrt{29}}{2})\cup(\frac{-1+\sqrt{29}}{2},+\infty)$,

f is concave on $(\frac{-1-\sqrt{29}}{2},\frac{-1+\sqrt{29}}{2})$.

Solution to d). Before we draw the graph of the function we note that $f(x)=0$ at $x=\frac{-5-\sqrt{21}}{2}$ and intersect the vertical axis at the point $y=1$.



Exercise 6.

a) Find y as a function of x by solving the following initial value problem

$$\begin{cases} y\frac{dy}{dx}=-\frac{9a^2}{(a+x)^2}, \\ y(0)=3\sqrt{a}, \end{cases}$$

where a is a positive constant.

b) Find $y'(0)$ and $y''(0)$ without solving the initial value problem and write the Taylor polynomial $P_2(x)$ around $x = 0$.

c) Find the quadratic approximate value of the solution of the initial value problem $y(x)$ at $x = a$.

d)What is error $E_2(x)$ at $x = a$?

Solution to a). We have $\int y\,dy = -9a^2 \int \frac{dx}{(a+x)^2}$. Integrating, we obtain

$$y^2 = \frac{18a^2}{x+a} + C \quad \Longrightarrow \quad y^2(0) = 18a + C = 9a \quad \Longrightarrow \quad C = -9a.$$

The solution to the initial value problem is given by $y^2 = \frac{18a^2}{x+a} - 9a$.

Solution to b). We have

$$y(0)y'(0) = \frac{-9a^2}{a^2} \quad \Longrightarrow \quad y'(0) = \frac{-9}{y(0)} = -\frac{3}{\sqrt{a}}.$$

We differentiate the equation $y\frac{dy}{dx} = -\frac{9a^2}{(a+x)^2}$ and find that $(y')^2 + yy'' = \frac{18a^2}{(x+a)^3}$. It implies

$$y''(0) = \frac{1}{y(0)}\Big(\frac{18}{a} - \big(y'(0)\big)^2\Big) \quad \Longrightarrow \quad y''(0) = \frac{3}{a\sqrt{a}}.$$

The second order Taylor polynomial around $x = 0$ is given by

$$P_2(x) = 3\sqrt{a} - \frac{3}{\sqrt{a}}x + \frac{3}{2a\sqrt{a}}x^2.$$

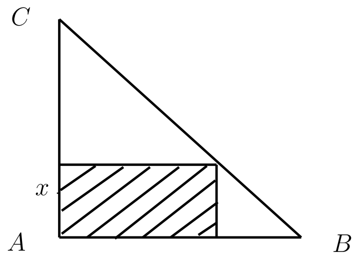
Solution to c). The quadratic approximate value of the function $y(x)$ at $x = a$ is given by $P_2(a) = \frac{3\sqrt{a}}{2}$.

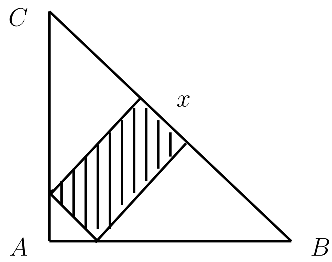
Solution to d). The error value is given by

$$E_2(a) = P_2(a) - y(a) = \frac{3\sqrt{a}}{2} - 0 = \frac{3\sqrt{a}}{2}.$$

Exercise 7.

Let ABC be a right triangle such that $|AB| = |AC|$ and $|BC| = 2\sqrt{2}$. Let the rectangle be inscribed to the triangle ABC in two ways as it is shown on the pictures.





In both cases find the length of the sides of the rectangles such that the areas are maximal. What is common in both cases and what is the difference?

Solution. By using the Pythagorus theorem we find that $|AB| = 2$. *The first case.* Let us denote the side of the rectangle by x, then the other side will be $2 - x$ and the area is the function of x given by

$$f(x) = x(2 - x).$$

The maximum is given by the zero of the derivative: $f'(x) = 2-2x = 0$ if $x = 1$. So the rectangle have to have equal sides and it will be the square.

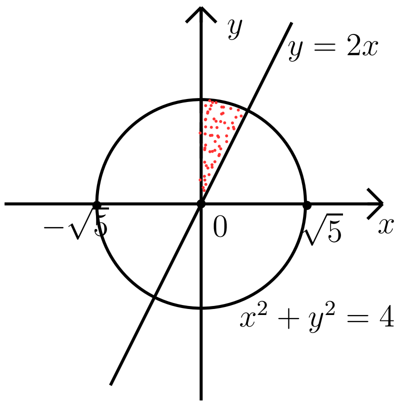
The second case. We denote by x the side of the rectangle that lies on the side BC of the triangle. Then the other side is equal to $\frac{1}{2}(2\sqrt{2}-x)$ and the area is given by the function $f(x) = \frac{1}{2}(2\sqrt{2} - x)x$. We find

$$f'(x) = \frac{1}{2}(2\sqrt{2} - 2x) = 0 \quad \text{if} \quad x = \sqrt{2}.$$

Thus the rectangle has the sides $\sqrt{2}$ and $\frac{1}{\sqrt{2}}$.
We see that the maximal areas are equal to 1, but in the first case the rectangle has equal sides, and in the second case they are different.

Exercise 8.

Calculate the volume of the body obtained by the rotation around the y-axis of the region inside of the circle $x^2+y^2=5$ between the straight line $y=2x$ and the vertical line. On the picture it is the dashed domain.



Solution. First we need to fund the point of intersection of the circle and the straight line.

$$x^2 + 4x^2 = 5 \quad \Longrightarrow \quad x = \pm 1.$$

Thus the intersection point that we are interested in is $x = 1$, $y = 2$. The volume of the body is given by the integral

$$\pi\Big(\int_0^2 \frac{y^2}{4}\,dy + \int_2^{\sqrt{5}} (5-y^2)\,dy\Big) = \frac{10\pi}{3}(\sqrt{5}-2).$$

Thanks for the exam!

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