FASIT - MAT121 V15

Oppgave 1.

1.1

$$\left[\begin{array}{cccc|c} 3 & -4 & -1 & 2 & -2 \\ 1 & 2 & 0 & -1 & 3 \\ 3 & -2 & -1 & 0 & 0 \\ 1 & 4 & 2 & a & b \end{array}\right] \sim \left[\begin{array}{cccc|c} 3 & -4 & -1 & 2 & -2 \\ 0 & 10 & 1 & -5 & 11 \\ 0 & 2 & 0 & -2 & 2 \\ 0 & 16 & 7 & 3a-2 & 3b+2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 3 & -4 & -1 & 2 & -2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 10 & 1 & -5 & 11 \\ 0 & 16 & 7 & 3a-2 & 3b+2 \end{array}\right]$$

$R_2 \to 3R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to 3R_4 - R_1$

$R_2 \leftrightarrow \frac{1}{2} R_3$

$$\sim \left[\begin{array}{cccc|c} 3 & -4 & -1 & 2 & -2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 7 & 3a+14 & 3b-14 \end{array}\right] \sim \left[\begin{array}{cccc|c} 3 & -4 & -1 & 2 & -2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 3a-21 & 3b-21 \end{array}\right]$$

$R_3 \to R_3 - 10R_2$
$R_4 \to R_4 - 16R_2$

$R_4 \to R_4 - 7R_3$

a) Systemet har ingen løsning når $3a-21=0$ og $3b-21 \neq 0$ fordi siste rekken er ekvivalent til $0=k$ der $k \neq 0$.

$\underline{\underline{a=7,\ b\neq 7}}$

b) Systemet har uendelig mange løsninger om det er frie parameter. Det skjer når $3a-21=0$ og $3b-21=0$ dvs

$\underline{\underline{a=7,\ b=7}}$

c) én løsning når alle kolonne er pivot kolonner, dvs $3a-21 \neq 0$

$\underline{\underline{a \neq 7}}$

1.2 $a=7,\ b=7$. x_1, x_2, x_3 ledende, x_4 fri.

Fra trappeformen til A:

$$\left[\begin{array}{cccc|c} 3 & -4 & -1 & 2 & -2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 + 5x_4 = 1 \quad \Rightarrow x_3 = 1 - 5x_4$

$x_2 - x_4 = 1 \quad \Rightarrow x_2 = 1 + x_4$

$3x_1 - 4x_2 - x_3 + 2x_4 = 2$

$3x_1 - 4 - 4x_4 - 1 + 5x_4 + 2x_4 = -2 \Rightarrow 3x_1 + 3x_4 = +3$

$x_1 = 1 - x_4$

generelle løsning

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-x_4 \\ 1+x_4 \\ 1-5x_4 \\ x_4 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ -5 \\ 1 \end{bmatrix}}}$$

1.3 $\text{Nul}\, A = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ -5 \\ 1 \end{bmatrix}\right\}$ dermed $\mathcal{B} = \left\{\begin{bmatrix} -1 \\ 1 \\ -5 \\ 1 \end{bmatrix}\right\}$ er en basis for Nul A

(en mengde som består av kun én vektor ulik $\underline{0}$ er alltid lin. uavh.).

1.4 Siden kolonnene 1, 2, 3 er pivot kolonner i trappeformen, kolonnene

$\underline{a}_1 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\underline{a}_2 = \begin{bmatrix} -4 \\ 2 \\ -2 \\ 4 \end{bmatrix}$, $\underline{a}_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ er en basis for Col A.

$$\underline{\underline{\mathcal{B} = \left\{\begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}\right\}}}$$

Vektoren $\underline{b} \in \text{Col}\, A$ når systemet $A\underline{x} = \underline{b}$ er konsistent (har løsning). Dette skjer når $b = 7$ (samme som under 1.1.b). Koord. til $\underline{b}$ er allerede regnet ut i 1.2, dvs $\underline{\underline{[\underline{b}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}}$

(1.5) $a=0$. $A = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 3 & -2 & -1 & 0 \\ 1 & 4 & 2 & 0 \end{bmatrix} \end{array}$ Broker Ko-factor formel mht Kol. 4.

$$\det A = -2 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 3 & -2 & -1 \\ 1 & 4 & 2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 3 & -4 & -1 \\ 3 & -2 & -1 \\ 1 & 4 & 2 \end{vmatrix} = -2 \cdot \left(1\,(\cancel{-4+4}) - 2\,(6+1)\right)$$

$$- 3\,(\cancel{-4+4}) - 4\,(6+1) + 1\,(12+2)$$

$$= +\cancel{28} - \cancel{28} + 14 = \underline{\underline{14}}$$

Oppgave 2

(2.1) $\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4}$ lin. uavh $\iff [\underline{a_1} \;\; \underline{a_2} \;\; \underline{a_3} \;\; \underline{a_4}]\,\underline{x} = \underline{0} \implies \underline{x} = \underline{0}$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & -4 \\ 1 & 1 & 4 & 1 \\ 1 & 0 & 2 & 4 \end{bmatrix} \sim \underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}}{\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 4 & 2 \\ 0 & -1 & 2 & 5 \end{bmatrix}} \sim \underset{R_4 \to R_4 + R_2}{\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 4 & 2 \end{bmatrix}} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Systemet har en fri variabel (x_4) dvs at $\underline{a_4}$ er lin. Komb. av $\underline{a_1}, \underline{a_2}, \underline{a_3}$

siden $\quad \underline{a_4} = 3\,\underline{a_1} - 4\,\underline{a_2} + \frac{1}{2}\,\underline{a_3}$

dermed $\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4}$ er <u>lin. avhengige.</u>

(2.2) $W = \text{span}\{\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4}\} = \text{Span}\{\underline{a_1}, \underline{a_2}, \underline{a_3}\}$

$\underline{a_1}, \underline{a_2}, \underline{a_3}$ er lin. uavhengige siden de tilsvarer pivot kolonner (se 2.1)
og spenner ut W.

$\mathcal{B} = \{\underline{a_1}, \underline{a_2}, \underline{a_3}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} \right\}$ er en basis for W.

dim W = 3 (antall basis vektorer).

(2.3) $\underline{u} \in W^\perp \iff \underline{u} \perp W = \text{Span}\{\underline{a_1}, \underline{a_2}, \underline{a_3}\} \iff \underline{u} \perp \underline{a_i}, \; i = 1,2,3$

siden $\{\underline{a_1}, \underline{a_2}, \underline{a_3}\}$ basis av W.

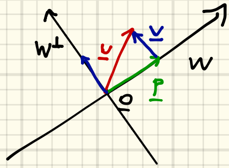
$\underline{u} \cdot \underline{a_1} = \begin{bmatrix} -4 \\ 1 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -4 + 1 + 4 - 1 = 0 \quad$ dermed $\underline{u} \perp \underline{a_1}$

$\underline{u} \cdot \underline{a_2} = \begin{bmatrix} -4 \\ 1 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = -4 + 2 + 4 = 2 \neq 0 \quad$ dermed $\underline{u} \not\perp \underline{a_2}$

$\underline{\underline{\underline{u} \notin W^\perp}}$

(2.4) Finn $\underline{u} = \text{proj}_W \underline{u} + \underline{v}$

$\quad = \underline{p} + \underline{v}$



For å finne $\underline{u} = \text{proj}_W \underline{u} + \underline{v}$ vi finner en ortogonale basis for W.

$\underline{v}_1 = \underline{a}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $\qquad \underline{v}_1 \cdot \underline{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 4$ $\qquad \underline{a}_2 \cdot \underline{v}_1 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 4$

$$\underline{v}_2 = \underline{a}_2 - \frac{\underline{a}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix} - \frac{4}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \qquad \underline{v}_2 \cdot \underline{v}_2 = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} = 2$$

$$\underline{v}_3 = \underline{a}_3 - \frac{\underline{a}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 - \frac{\underline{a}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2 = \begin{bmatrix}0\\2\\4\\2\end{bmatrix} - \frac{\overset{2}{\cancel{8}}}{\cancel{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{0}{2}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

$$= \begin{bmatrix}-2\\0\\2\\0\end{bmatrix}$$

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ er en ortogonale basis for W.

$$\underline{p} = \text{proj}_W \underline{u} = \frac{\underline{u} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 + \frac{\underline{u} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2 + \frac{\underline{u} \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3} \underline{v}_3$$

$\underline{u} \cdot \underline{v}_1 = 0 \qquad \underline{u} \cdot \underline{v}_2 = \begin{bmatrix}-4\\1\\4\\-1\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\\-1\end{bmatrix} = 2 \qquad \underline{u} \cdot \underline{v}_3 = \begin{bmatrix}-4\\1\\4\\-1\end{bmatrix} \cdot \begin{bmatrix}-2\\0\\2\\0\end{bmatrix} = 16$

$$\underline{p} = \frac{\cancel{2}}{\cancel{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} + \frac{\overset{2}{\cancel{16}}}{\cancel{8}}\begin{bmatrix}-2\\0\\2\\0\end{bmatrix} = \begin{bmatrix}-4\\1\\4\\-1\end{bmatrix} = \underline{u} \qquad (\text{dermed } \underline{u} \in W)$$

$\underline{v} = \underline{u} - \underline{p} = \underline{0}$

$$\underline{\underline{\text{proj}_W \underline{u} = \underline{u}, \ \underline{v} = \underline{0}}}$$

(2.5) Vi har allerede funnet en ortogonale basis i 2.4. For å finne en ortonormale basis trenger vi kun å normalisere vektorene $\underline{v}_1, \underline{v}_2, \underline{v}_3$.

$$\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad \underline{u}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \qquad \underline{u}_3 = \frac{\underline{v}_3}{\|\underline{v}_3\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

$$\underline{\underline{\mathcal{C} = \left\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\right\}}}$$ er den ønskede ortonormale basis.

Oppgave 3

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

3.1 Vi ser at $\det A = 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \neq 0$ dermed A inverterbar

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 2/3 \end{array}\right]$$

$R_3 \to 2R_3 - R_2$ $\qquad$ $R_3 \to R_3/3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 4/3 & -2/3 \\ 0 & 0 & 1 & 0 & -1/3 & 2/3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2/3 & -1/3 \\ 0 & 0 & 1 & 0 & -1/3 & 2/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/3 & -1/3 \\ 0 & -1/3 & 2/3 \end{bmatrix}$$

3.2 Egenverdiene: $0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} =$

$$= (1-\lambda)\left((2-\lambda)^2 - 1\right) = (1-\lambda)(\lambda^2 - 4\lambda + 3) = (1-\lambda)(\lambda - 1)(\lambda - 3)$$

egenverdiene er $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 3$.

A er diagonaliserbar siden den er symmetrisk (alle symm. matriser er faktisk <u>ortogonalt</u> diagonaliserbare!).

Egenvektorene: $\lambda = 1$:

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 ledende, x_1, x_3 frie

$$\underline{v} = \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

vi har 2 egenvektorer: $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\lambda = 3$:

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 fri
$x_2 = x_3$
$x_1 = 0$

$$\rightarrow \underline{x} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

dette gir egenvektor $\underline{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Diagonalisering: $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix}$

$A = PDP^{-1}$

3.3 $Q(\underline{x}) = x_1^2 + 2x_2^2 + 2x_2x_3 + 2x_3^2$

$$= \underline{x}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \underline{x}$$ med samme matrise A som over.

Formen er $\underline{\underline{\text{positiv definert}}}$ siden egenverdiene til A er positive.

For å diagonalisere formen, vi må finne en ortogonal diagonalisering av A.

Vi ser at $\underline{v}_1, \underline{v}_2, \underline{v}_3$ er ortogonale
$\underline{v}_1 \cdot \underline{v}_2 = 0$,
$\underline{v}_1 \cdot \underline{v}_3 = 0$
$\underline{v}_2 \cdot \underline{v}_3 = 0$

men ikke normale, siden $\|\underline{v}_2\| = \sqrt{2}$
$\|\underline{v}_3\| = \sqrt{2}$

Ved å ta $\tilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ vi har at variabelskiftet

$\underline{y} = \tilde{P}^T \underline{x}$ diagonaliserer formen. $\underline{\underline{Q(\underline{y}) = y_1^2 + y_2^2 + 3y_3^2.}}$

(3.4) $T(\underline{e}_1) = \underline{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $T(\underline{e}_2) = 2\underline{e}_2 + \underline{e}_3 = \begin{bmatrix}0\\2\\1\end{bmatrix}$, $T(\underline{e}_3) = \underline{e}_2 + 2\underline{e}_3 = \begin{bmatrix}0\\1\\2\end{bmatrix}$

for enhver $\underline{x} \in \mathbb{R}^3$ $\quad \underline{x} = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3$

$T(\underline{x}) = x_1 T(\underline{e}_1) + x_2 T(\underline{e}_2) + x_3 T(\underline{e}_3)$ siden T lineær.

$$= \begin{bmatrix} T(\underline{e}_1) & T(\underline{e}_2) & T(\underline{e}_3) \end{bmatrix} \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&2&1\\0&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Standardmatrisen: $\underline{\underline{\begin{bmatrix}1&0&0\\0&2&1\\0&1&2\end{bmatrix}}}$

$$T\left(\begin{bmatrix}3\\2\\1\end{bmatrix}\right) = A\begin{bmatrix}3\\2\\1\end{bmatrix} = \underline{\underline{\begin{bmatrix}3\\5\\4\end{bmatrix}}}$$

(3.5) Vi ønsker å regne ut $A\underline{x}$ via diagonalisering av A

Her kan vi bruke både $A = PDP^{-1}$ (vanlig diagonalisering) eller
$A = \tilde{P}D\tilde{P}^T$ (ortog. diagonalisering, siden A er symmetrisk).

Det er lettere å bruke $A = \tilde{P}D\tilde{P}^T$ siden vi slipper å måtte regne ut P^{-1}!

Diag. diagramm:

$$\begin{tikzcd}
\underline{x} \arrow[r, "\text{mult}"', "\text{m/ } A"'] \arrow[d, "\text{mult m/ } \tilde{P}^T"'] & A\underline{x} = P(D(\tilde{P}\underline{x})) \\
\tilde{P}^T\underline{x} = [\underline{x}]_B \arrow[r, "\text{mult. m/}", "D"'] & D[\underline{x}]_B = D(\tilde{P}^T\underline{x}) \arrow[u, "\text{mult. m/ } \tilde{P}"']
\end{tikzcd}$$

$B \equiv$ ortonorm. egenvektor basis.

$$[\underline{x}]_B = \tilde{P}^T\underline{x} = \begin{bmatrix}1&0&0\\0&-\sqrt{2}/2&\sqrt{2}/2\\0&\sqrt{2}/2&\sqrt{2}/2\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}3\\-\sqrt{2}/2\\3\sqrt{2}/2\end{bmatrix}$$

(merk: $\tilde{P}$ symm.)

$$D[\underline{x}]_B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 \\ -\sqrt{2}/2 \\ 9\sqrt{2}/2 \end{bmatrix}$$

$$\vec{P}\left(D[\underline{x}]_B\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3 \\ -\sqrt{2}/2 \\ 9\sqrt{2}/2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}}}$$

$$\frac{2}{4} + \frac{18}{4} = \frac{20}{4} = 5$$

$$-\frac{2}{4} + \frac{18}{4} = \frac{16}{4} = 4$$

Som forventet fra deloppgave 3.4.

Oppgave 4.

4.1

k	1	2	3	4	5
y	1	4	4	5	8

lin. modell: $y(k) = a + bk$

$k=1$: $a + b\cdot 1 = 1$
$k=2$: $a + b\cdot 2 = 4$
$\vdots$
$k=5$: $a + b\cdot 5 = 8$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 4 \\ 5 \\ 8 \end{bmatrix}}_{\underline{b}}$$

Minste-Kvadrater problem $A\underline{x} = \underline{b}$

Normale ligninger: $A^T A \underline{x} = A^T \underline{b}$

$$A^T A = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \qquad A^T \underline{b} = \begin{bmatrix} 22 \\ 81 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} -1/10 \\ 3/2 \end{bmatrix}$$

$\underline{\underline{y(k) = -\frac{1}{10} + \frac{3}{2}k}}$ er den minste-kvadraters løsning.

(ca. 1.5 poeng per kamp i gjennomsnitt).

$k=8 \rightarrow y(8) = \frac{119}{10} \sim \underline{\underline{12 \text{ poeng}}}$

$k=30 \rightarrow y(30) = \frac{449}{10} \sim \underline{\underline{45 \text{ poeng}}}$

4.2 a) $q(x) = q_0 + \cdots + q_m x^m$.

$A = X \Lambda X^{-1}$ $\quad A^k = X \Lambda^k X^{-1}$ $\quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$

$$q(A) = q_0 I + q_1 A + \cdots + q_m A^m$$
$$= q_0 I + q_1 X \Lambda X^{-1} + \cdots + q_m X \Lambda^m X^{-1}$$
$$= X (q_0 I + q_1 \Lambda + \cdots + q_m \Lambda^m) X^{-1}$$
$$= X \, q(\Lambda) \, X^{-1}.$$

der $q(\Lambda) = \begin{pmatrix} q(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & q(\lambda_m) \end{pmatrix}$

q.e.d.

b) Anta $p(\lambda) = \det(A - \lambda I)$

siden A diagonaliserbar: $\quad p(A) = X p(\Lambda) X^{-1}$ (se pkt a)).

Men $p(\Lambda) = \begin{pmatrix} p(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & p(\lambda_m) \end{pmatrix}$ og $p(\lambda_i) = 0$ siden λ_i egenverdi av A dermed rot av $p(\lambda) = \det(A - \lambda I)$.

Dermed: $\quad p(\Lambda) = 0$

$$p(A) = X \cdot 0 \cdot X^{-1} = 0 \qquad \text{q.e.d.}$$

c) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

Cayley-Hamilton: $p(A) = 0$ dermed $A^2 + I = 0$ og $A^2 = -I$.

Påstand: $A^{2k} = (-1)^k I \qquad A^{2k+1} = (-1)^k A$

$k=0$: $\quad A^0 = I = (-1)^0 I$ ✓ $\qquad A^1 = (-1)^0 A = A$

Anta påstanden sant for k;

$A^{2(k+1)} = A^{2k+2} = A^{2}(A^{2k}) = (-I)(-1)^{k} I = (-1)^{k+1} I$ ✓

$A^{2(k+1)+1} = A^{2k+1+2} = A^{2} A^{2k+1} = (-I)(-1)^{k} A = (-1)^{k+1} A$ ✓

Det medfører at påstanden er sant også for k+1, dermed for alle

k. Q.E.D.