English

University of Bergen

The Faculty of Mathematics and Natural Sciences

Exam in MAT121 - Linear algebra

September 26, 2019, from 09.00 to 13.00

The exam consists of two parts:

Exercises 1-20 is of type "multiple choice". You have to choose the correct answer and mark it. In exercise 20 the questions can have several correct answers. This part assumes that you give answers on the computer.

The exercises 21-22 require from you an ability to make a proof of some statements. If you have difficulty to write it on the computer, just write it by hand on the additional ark and deliver to the Inspera system as a pdf file.

1.3 The augmented matrix

$$\begin{bmatrix} 3 & -4 & 2 & a \\ -9 & 12 & -6 & b \\ -6 & 8 & -4 & c \end{bmatrix}$$

corresponds to a consistent system if (choose the correct answer)

- ○ $2a + c = 0$, $3a + b \neq 0$
- ○ $a + b + c = 0$
- ○ $2a + c = 3a + b$, and $a \neq 0$
- • $2a + c = 0$ and $3a + b = 0$
- ○ non of them

2.3 A linear transformation $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by a rotation on some angle in the counterclockwise direction. Let Φ rotate the vector $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ to the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then the matrix that corresponds to Φ is given by (choose the correct answer)

○ $\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

○ $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

○ $\begin{bmatrix} \cos(\frac{\pi}{6}) & \sin(\frac{\pi}{6}) \\ -\sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{bmatrix}$

● $\begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}$

○ non of them

3.3 Let $\alpha = (\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \overrightarrow{\alpha}_3, \overrightarrow{\alpha}_4)$ where

$$\overrightarrow{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{\alpha}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{\alpha}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \overrightarrow{\alpha}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by

$$T(\overrightarrow{\alpha}_1) = \overrightarrow{\beta}_1, \quad T(\overrightarrow{\alpha}_2) = \overrightarrow{\beta}_2, \quad T(\overrightarrow{\alpha}_3) = \overrightarrow{\beta}_3, \quad T(\overrightarrow{\alpha}_4) = \overrightarrow{\beta}_4,$$

where the vectors $\overrightarrow{\beta}_j$, $j = 1, 2, 3, 4$ are given by

$$\overrightarrow{\beta}_1 = \overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2, \qquad \overrightarrow{\beta}_2 = \overrightarrow{\alpha}_2 + \overrightarrow{\alpha}_3,$$
$$\overrightarrow{\beta}_3 = \overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4, \qquad \overrightarrow{\beta}_4 = \overrightarrow{\alpha}_4 + \overrightarrow{\alpha}_1.$$

Then the standard matrix of the transformation T is given by (choose the correct answer)

○ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

- ○ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$
- ○ $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
- ● non of them

4.3 Let vectors $\overrightarrow{\gamma}_1, \overrightarrow{\gamma}_2, \overrightarrow{\gamma}_3, \overrightarrow{\gamma}_4$ are given by

$$\overrightarrow{\gamma}_1 = \overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_3, \qquad \overrightarrow{\gamma}_2 = -\overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2 + \overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4,$$

$$\overrightarrow{\gamma}_3 = \overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4, \qquad \overrightarrow{\gamma}_4 = 2\overrightarrow{\alpha}_2 + 4\overrightarrow{\alpha}_3 + 2\overrightarrow{\alpha}_4,$$

where $\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \overrightarrow{\alpha}_3, \overrightarrow{\alpha}_4$ is a basis of a vector space V. The dimension of $W = \text{span}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is equal to (choose the correct answer)

- ○ 1
- ○ 2
- ● 3
- ○ 4
- ○ 0

5.3 Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by

$$T(\overrightarrow{e}_1) = \overrightarrow{e}_1 + \overrightarrow{e}_3, \quad T(\overrightarrow{e}_2) = -\overrightarrow{e}_1 + \overrightarrow{e}_2 + \overrightarrow{e}_3 + \overrightarrow{e}_4,$$

$$T(\overrightarrow{e}_3) = \overrightarrow{e}_1 - \overrightarrow{e}_3 + \overrightarrow{e}_4, \quad T(\overrightarrow{e}_4) = 2\overrightarrow{e}_2 + 4\overrightarrow{e}_3 + 2\overrightarrow{e}_4,$$

in standard basis $\overrightarrow{e}_j$, $j = 1, 2, 3, 4$. Then the dimension of the null space is (choose the correct answer)

- ● 1
- ○ 2
- ○ 3
- ○ 4
- ○ 0

6.3 Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given in the problem 5.3 Then the basis of the null space of the transformation is (choose the correct answer)

- ● $\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

- ○ $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

- ○ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

- ○ $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \end{bmatrix}$

- ○ non of them

7.3 The determinant of the matrix

$$\begin{bmatrix} 2 & 1 & 2 \\ 2-4\log 8 & -1 & 2+4\log 8 \\ 1-\log 8 & 3 & 1+\log 8 \end{bmatrix}$$

is equal to (choose the correct answer)

- ○ $\log 8$
- ● $-48\log 8$
- ○ $-\log 8$
- ○ $48\log 8$
- ○ $-4(\log 8)^2$
- ○ non of them

8.3 Let S be the parallelogram determined by the vectors

$$\vec{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$. The area of the image of S under the mapping $\vec{x} \mapsto A\vec{x}$ is equal to (choose the correct answer)

○ -12
○ -4
○ 4
○ 21
● 12
○ non of them

9.3 Let

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 2 \end{bmatrix}.$$

The inverse to adjugate matrix $(\text{adj}A)^{-1}$ is given by (choose the correct answer)

○ $\begin{bmatrix} 1 & -2 \\ -1 & 5/2 \end{bmatrix}$

○ $\begin{bmatrix} 2 & -4 \\ -2 & 5 \end{bmatrix}$

● $\begin{bmatrix} 5/2 & 1 \\ 2 & 1 \end{bmatrix}$

○ $\begin{bmatrix} 4 & -8 \\ -4 & 10 \end{bmatrix}$

○ $\begin{bmatrix} -10 & -4 \\ -8 & -4 \end{bmatrix}$

○ non of them

10.3 Let

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \vec{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

be two bases in $\mathbb{R}^2$. The change-of-basis matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{C}}$ is equal to (choose the correct answer)

- ○ $\begin{bmatrix} -3 & -2 \\ 4 & -3 \end{bmatrix}$
- ○ $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$
- • $\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$
- ○ $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
- ○ $\begin{bmatrix} -1 & 1 \\ 8 & -5 \end{bmatrix}$
- ○ non of them

11.3 Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The eigenvalues of A are given by (choose the correct answer)

- ○ $\lambda = -2$
- ○ $\lambda = -2, -2, 0$
- ○ $\lambda = 2$
- • $\lambda = 0, 2$
- ○ non of them

12.3 Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The eigenvectors of A are given by (choose the correct answer)

- ○ $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

• $\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

○ non of them

13.3 Let $A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$. It is known that $\lambda = 3, 1$ are among the eigen values of A. The diagonalization of the matrix A is given by (choose the correct answer)

○ $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

○ $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

○ $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

• $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

○ The matrix A is not diagonalisable

14.3 Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}$. The matrix $(A^T A)^{-1}$ is given by (choose the correct answer)

- ○ $\begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix}$
- ○ $\begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$
- ○ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- ● $\frac{1}{54} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
- ○ non of these

15.3 The least square solution to the problem $A\overrightarrow{x} = \overrightarrow{b}$, where

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}, \qquad \overrightarrow{b} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is given by (choose the correct answer)

- ○ $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}$
- ○ $\begin{bmatrix} 54 \\ 27 \\ 27 \end{bmatrix}$

○ $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

● $\frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

○ non of these

16.3. The quadratic form $Q = 3x_1^2 + 4x_1x_2$ is (choose the correct answer)

○ positive definite
○ negative definite
● indefinite
○ non of them

17.3 The matrix P that orthogonally diagonalises the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ is given by (choose the correct answer)

○ $\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$

○ $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

● $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

○ $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$

○ The matrix A is not diagonalisable

18.3 The orthogonal complement to $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$, is given by (choose the correct answer)

- ○ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- ○ $\text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$
- • $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \right\}$
- ○ $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$
- ○ non of these

19.3 The distance between points $p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $q = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is equal to (choose the correct answer)

- ○ $-2\sqrt{2}$
- • $2\sqrt{2}$
- ○ 2
- ○ 4
- ○ -2
- ○ non of these

20.3 A square $(n \times n)$-matrix A is invertible, if (choose the correct answer. It can be several correct answers.)

- ○ $a_{11} + a_{22} + \ldots + a_{nn} \neq 0$
- • $\det A \neq 0$
- • The columns of the matrix A are linearly independent
- ○ The eigen values of the matrix A are all different
- ○ $A^T = -A$
- ○ The matrix A is symmetric
- ○ The matrix has n eigen vectors
- • The rank of the matrix is equal to n
- • The rows of the matrix A are linearly independent
- ○ There is a matrix B such that $AB = BA$

- Null$(A) = \{0\}$
- The matrix A^T is invertible
- ◦ There is an invertible matrix C such that $CAC^{-1} = I$

21.3. Suppose that A and B are symmetric matrices and $AB = BA$. Is the matrix AB orthogonally diagonalizable? Explain why.

We use the theorem that states that a matrix is diagonalizable if and only if it is symmetric. So to show that AB orthogonally diagonalizable it is enough to show that AB is symmetric. We calculate

$$(AB)^T = B^T A^T = BA = AB$$

So the matrix AB is symmetric and therefore orthogonally diagonalizable.

22.3. Suppose $A = QR$, where R is an invertible matrix. Show that A and Q have the same column space.

Let A be a $m \times n$ matrix and $\overrightarrow{x} \in \mathrm{Col}(A)$, that is $\overrightarrow{x} = A\overrightarrow{y}$ for some $\overrightarrow{y} \in \mathbb{R}^n$. Then

$$\overrightarrow{x} = A\overrightarrow{y} = QR\overrightarrow{y} = Q\overrightarrow{z},$$

where we denote by $\overrightarrow{z} = R\overrightarrow{y} \in \mathbb{R}^n$. The vector $\overrightarrow{z}$ belong to $\mathbb{R}^n$, since R is invertible $n \times n$ matrix. So, $\overrightarrow{x} \in \mathrm{Col}(Q)$.

Conversely, let $\overrightarrow{x} \in \mathrm{Col}(Q)$, that is $\overrightarrow{x} = Q\overrightarrow{y}$ for some $\overrightarrow{y} \in \mathbb{R}^n$. Then since R is invertible we obtain that the equation $R^{-1}\overrightarrow{y} = \overrightarrow{w}$ has the unique solution

$$\overrightarrow{y} = R\overrightarrow{w}$$

for any $\overrightarrow{w} \in \mathbb{R}^n$. Thus

$$\overrightarrow{x} = Q\overrightarrow{y} = QR\overrightarrow{w} = A\overrightarrow{w}, \quad \overrightarrow{w} \in \mathbb{R}^n.$$

We conclude that $x \in \mathrm{Col}(A)$.