English

University of Bergen

The Faculty of Mathematics and Natural Sciences

Solutions to the Exam in MAT121 - Linear algebra

May 13, 2019, from 09.00 to 13.00

The exam consists of two parts:

Exercises 1-20 is of type "multiple choice". You have to choose the correct answer and mark it. In exercise 20 the questions can have several correct answers. This part assumes that you give answers on the computer.

The exercises 21-22 require from you an ability to make a proof of some statements. If you have difficulty to write it on the computer, just write it by hand on the additional ark and deliver to the Inspera system as a pdf file.

1.1 The augmented matrix

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 & a \\ 0 & 0 & -8 & -8 & 4 & b \\ 1 & -7 & -4 & 2 & 7 & c \end{bmatrix}$$

corresponds to a consistent system if (choose the correct answer)

- ○ $a \neq 0$, $b \neq 0$, $c = 0$
- ○ $a + 2b - c = 0$
- ○ $2b \neq 0$, and $a - c = 0$
- • $2(a - c) + b = 0$
- ○ non of them

After making echelon form of the matrix, we find that the operation with rows $2(R1 - R3) + R2$ produce the last row that contains only zeros.

2.1 A linear transformation $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by a rotation on some angle in the counterclockwise direction. Let Φ rotate the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to the vector $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Then the matrix that corresponds to Φ is given by (choose the correct answer)

- ○ $\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$
- ○ $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$
- ○ $\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix}$
- ● $\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$
- ○ non of them

The rotation matrix has the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ **with** $a^2 + b^2 = 1$**. Solving the equation**

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

we find the solution.

3.1 Let $\alpha = (\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \overrightarrow{\alpha}_3, \overrightarrow{\alpha}_4)$ where

$$\overrightarrow{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{\alpha}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \overrightarrow{\alpha}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{\alpha}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by

$T(\overrightarrow{\alpha}_1) = \overrightarrow{\beta}_1 + \overrightarrow{\beta}_2, \quad T(\overrightarrow{\alpha}_2) = \overrightarrow{\beta}_2 + \overrightarrow{\beta}_3, \quad T(\overrightarrow{\alpha}_3) = \overrightarrow{\beta}_3 + \overrightarrow{\beta}_4, \quad T(\overrightarrow{\alpha}_4) = \overrightarrow{\beta}_4 + \overrightarrow{\beta}_1,$

where the vectors $\overrightarrow{\beta}_j$, $j = 1, 2, 3, 4$ are given by

$$\overrightarrow{\beta}_1 = \overrightarrow{\alpha}_1, \qquad \overrightarrow{\beta}_2 = \overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2,$$
$$\overrightarrow{\beta}_3 = \overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2 + \overrightarrow{\alpha}_3, \qquad \overrightarrow{\beta}_4 = \overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2 + \overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4.$$

Then the standard matrix of the transformation T is given by (choose the correct answer)

- ○ $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

- ○ $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

- ○ $\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \right)^T$

- ● $\begin{bmatrix} -1 & 5 & -2 & 3 \\ -1 & 5 & -2 & 2 \\ 0 & 2 & 0 & 1 \\ -1 & 4 & -2 & 3 \end{bmatrix}$

- ○ $\begin{bmatrix} 1 & -1 & 0 & -1 \\ -2 & -2 & 0 & -2 \\ 5 & 5 & 2 & 4 \\ 3 & 2 & 1 & 3 \end{bmatrix}$

- ○ non of them

4.1 Let vectors $\overrightarrow{\gamma}_1, \overrightarrow{\gamma}_2, \overrightarrow{\gamma}_3, \overrightarrow{\gamma}_4$ are given by

$$\overrightarrow{\gamma}_1 = \overrightarrow{\alpha}_2 - \overrightarrow{\alpha}_3, \qquad \overrightarrow{\gamma}_2 = \overrightarrow{\alpha}_1 + 2\overrightarrow{\alpha}_4,$$

$$\overrightarrow{\gamma}_3 = \overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2 - \overrightarrow{\alpha}_3 + 2\overrightarrow{\alpha}_4, \qquad \overrightarrow{\gamma}_4 = 2\overrightarrow{\alpha}_1 + 2\overrightarrow{\alpha}_2 - 2\overrightarrow{\alpha}_3 + 4\overrightarrow{\alpha}_4,$$

where $\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \overrightarrow{\alpha}_3, \overrightarrow{\alpha}_4$ is a basis of a vector space V. The dimension of $W = \text{span}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is equal to (choose the correct answer)

- ○ 1
- ● 2
- ○ 3
- ○ 4
- ○ 0

5.1 Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by

$$T(\overrightarrow{e}_1) = \overrightarrow{e}_2 - \overrightarrow{e}_3, \quad T(\overrightarrow{e}_2) = \overrightarrow{e}_1 + 2\overrightarrow{e}_4,$$
$$T(\overrightarrow{e}_3) = \overrightarrow{e}_1 + \overrightarrow{e}_2 - \overrightarrow{e}_3 + 2\overrightarrow{e}_4, \quad T(\overrightarrow{e}_4) = 2\overrightarrow{e}_1 + 2\overrightarrow{e}_2 - 2\overrightarrow{e}_3 + 4\overrightarrow{e}_4,$$

Then the dimension of the null space is (choose the correct answer)

- ○ 1
- ● 2
- ○ 3
- ○ 4
- ○ 0

6.1 Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given in the problem 5.1 Then the basis of the null space of the transformation is (choose the correct answer)

- ○ $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- ● $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$
- ○ $\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ -2 \\ 1 \end{bmatrix}$
- ○ $\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$
- ○ $\begin{bmatrix} -3 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -6 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \end{bmatrix}$
- ○ non of them

7.1 The determinant of the matrix

$$\begin{bmatrix} 1+3\sqrt{10} & 1-3\sqrt{10} & 1 \\ 2\sqrt{10} & -2\sqrt{10} & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

is equal to (choose the correct answer)

- ○ $20\sqrt{10}$
- ● $-20\sqrt{10}$
- ○ $\sqrt{10}$
- ○ $10\sqrt{10}$
- ○ $-\sqrt{10}$
- ○ non of them

8.1 Let S be the parallelogram determined by the vectors

$$\overrightarrow{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}, \qquad \overrightarrow{b}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 5 & 2 \\ -15 & -6 \end{bmatrix}$. The area of the image of S under the mapping $\overrightarrow{x} \mapsto A\overrightarrow{x}$ is equal to (choose the correct answer)

- ○ 320
- ○ 25
- ○ 5
- ● 0
- ○ non of them

9.1 Let

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

The inverse to adjugate matrix $(\text{adj}A)^{-1}$ is given by (choose the correct answer)

- ○ $\begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$
- ○ $\begin{bmatrix} -10 & 6 \\ 8 & -4 \end{bmatrix}$
- ● $\begin{bmatrix} -1 & -3/2 \\ -2 & -5/2 \end{bmatrix}$
- ○ $\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$
- ○ $\begin{bmatrix} -1 & -3/2 \\ -2 & -5/2 \end{bmatrix}$
- ○ non of them

10.1 Let

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \vec{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$$

be two bases in $\mathbb{R}^2$. The change-of-basis matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{C}}$ is equal to (choose the correct answer)

- ○ $\begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$
- ○ $\begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$
- ● $\begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$
- ○ $\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}$
- ○ $\begin{bmatrix} -3/2 & -2 \\ -5/2 & 3 \end{bmatrix}$
- ○ non of them

11.1 Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$. The eigenvalues of A are given by (choose the correct answer)

- ○ $\lambda = -2, 1$
- ○ $\lambda = 2, -1$
- ○ $\lambda = 0, 1, -2$
- • $\lambda = 0, 2, -1$
- ○ non of them

12.1 Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$. The eigenvectors of A are given by (choose the correct answer)

- ○ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$
- ○ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$
- ○ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$
- • $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$
- ○ non of them

13.1 Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. It is known that $\lambda = 2, 8$ are among the eigen values of A. The diagonalization of the matrix A is given by (choose the correct answer)

- ○ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$
- ○ $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$
- ○ $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
- ● $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
- ○ The matrix A is not diagonalisable

14.1 Let $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$. The matrix $(A^T A)^{-1}$ is given by (choose the correct answer)

- ○ $\begin{bmatrix} 1 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix}$
- ○ $\begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}$
- ○ $\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$
- ● $\frac{1}{14} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$
- ○ non of these

15.1 The least square solution to the problem $A\overrightarrow{x} = \overrightarrow{b}$, where

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, \quad \overrightarrow{b} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

is given by (choose the correct answer)

- ○ $\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$
- ○ $\frac{1}{7}\begin{bmatrix} 42 \\ 14 \end{bmatrix}$
- ○ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- ● $\frac{1}{7}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- ○ non of these

16.1. The quadratic form $Q = 2x_1^2 + 6x_1x_2 - 6x_2^2$ is (choose the correct answer)

- ○ positive definite
- ○ negative definite
- ○ positive semidefinite
- ○ negative semidefinite
- ● non of them

17.1 The matrix P that orthogonally diagonalises the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ is given by (choose the correct answer)

- ○ $\begin{bmatrix} -1 & -3 \\ -3 & 1 \end{bmatrix}$
- ○ $\frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$

- ○ $\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$
- ● $\frac{1}{\sqrt{10}} \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$
- ○ The matrix A is not diagonalisable

18.1 The orthogonal complement to $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, is given by (choose the correct answer)

- ○ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- ○ $\text{span}\left\{ \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right\}$
- ● $\text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$
- ○ $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
- ○ non of these

19.1 The distance between points $p = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $q = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$ is equal to (choose the correct answer)

- ○ -4
- ● 4
- ○ 2
- ○ -2
- ○ $2\sqrt{3}$
- ○ non of these

20.1. Let A be $(m\times n)$-matrix, where. The equation $A\overrightarrow{x} = \overrightarrow{b}$ has a unique least square solution, if (choose the correct answer. It can be several correct answers.)

- ○ A^TA is a square matrix
- • A^TA is invertible
- • The columns of the matrix A are linearly independent
- ○ The eigen values of the matrix A^TA are all different
- • There no zero eigen values of the matrix A^TA
- ○ $A^TA = AA^T$
- ○ The matrix A^TA is symmetric
- • $\dim(\text{Null}A) = 0$
- ○ $\overrightarrow{b} \in \text{Col}(A)$
- ○ The matrix A^TA has n eigen vectors
- • The rank of the matrix A is equal to n
- ○ $\overrightarrow{b} \in \text{Col}(A^TA)$
- • $\dim \text{Rad}(A) = n$

21.1. Let A be an $m \times n$ matrix. Prove that every vector $\overrightarrow{x}$ in $\mathbb{R}^n$ can be written in the form $\overrightarrow{x} = \overrightarrow{p} + \overrightarrow{u}$, where $p \in \text{Row}(A)$ and $u \in \text{Null}(A)$. Also show that if the equation $A\overrightarrow{x} = \overrightarrow{b}$ is consistent, then there is a unique $\overrightarrow{p} \in \text{Row}(A)$, such that $A\overrightarrow{p} = \overrightarrow{b}$.

Let $\overrightarrow{y} \in \mathbb{R}^m$**, then** $\overrightarrow{p} = A^T\overrightarrow{y} \in \text{Row}(A)$**. Let** $\overrightarrow{u} \in \text{Null}(A)$**, i.e.** $A\overrightarrow{u} = \overrightarrow{0}$ **then**

$$\overrightarrow{p} \cdot \overrightarrow{u} = \overrightarrow{p}^T\overrightarrow{u} = \left(A^T\overrightarrow{y}\right)^T\overrightarrow{u} = \overrightarrow{y}^T(A^T)^T\overrightarrow{u} = \overrightarrow{y}^T(A\overrightarrow{u}) = 0.$$

It shows that $\text{Row}(A)$ **is orthogonal to** $\text{Null}(A)$**. By the rank theorem we also know that**

$$\text{rank}(A) + \dim\left(\text{Null}(A)\right) = \dim\left(\text{Row}(A)\right) + \dim\left(\text{Null}(A)\right) = n.$$

It shows that

$$\text{Null}(A) = \Big(\text{Row}(A)\Big)^{\perp}$$

Thus any vector $\overrightarrow{x} \in \mathbb{R}^n$ **can be written as a sum of a unique (!) vector** $\overrightarrow{p} = \text{proj}_{\text{Row}(A)}\overrightarrow{x} \in \text{Row}(A)$ **and a vector** $\overrightarrow{u} \in \Big(\text{Row}(A)\Big)^{\perp} = \text{Null}(A)$**.**

If the equation $A\overrightarrow{x} = \overrightarrow{b}$ **is consistent, then there is at least one** $\overrightarrow{x} \in \mathbb{R}^n$ **such that** $A\overrightarrow{x} = \overrightarrow{b}$**. For that** $\overrightarrow{x}$ **we find unique** $\overrightarrow{p} \in \text{Row}(A)$ **such that**

$$\overrightarrow{x} = \overrightarrow{p} + \overrightarrow{u}.$$

Then

$$A\vec{x} = A(\vec{p} + \vec{u}) = A\vec{p} = \vec{b},$$

since $A\vec{u} = 0$**. So, there is a unique** $\vec{p} \in \mathrm{Row}(A)$ **such that** $A\vec{p} = \vec{b}$**.**

22.1. Suppose A is $m \times n$ matrix with linearly independent columns. Let $\vec{b} \in \mathbb{R}^m$. Use the normal equation to produce a formula for $\hat{\vec{b}} = \mathrm{proj}_{\mathrm{Col(A)}}\vec{b}$. The formula does not require an orthogonal basis for $\mathrm{Col}(A)$.

From the normal equation we know that in the case when the matrix A **has linearly independent columns, the square matrix** A^TA **is invertible and therefore the least square problem has (unique) solution** $\hat{x}$ **given by**

$$\vec{\hat{x}} = (A^TA)^{-1}A^T\vec{b}.$$

From other side

$$A\vec{\hat{x}} = \hat{\vec{b}} = \mathrm{proj}_{\mathrm{Col(A)}}\vec{b}.$$

Thus

$$\hat{\vec{b}} = \mathrm{proj}_{\mathrm{Col(A)}}\vec{b} = A\vec{\hat{x}} = A(A^TA)^{-1}A^T\vec{b}$$

for any $\vec{b} \in \mathbb{R}^m$**.**

Additionally (it is not requered) one can show the statement: if the matrix A **has linearly independent columns, then the square matrix** A^TA **is invertible.**

Indeed if $A\vec{x} = \vec{0}$**, then** $A^TA\vec{x} = \vec{0}$**.**

The other way around: let us assume that $A^TA\vec{x} = \vec{0}$**, then**

$$\|A\vec{x}\|^2 = (A\vec{x}) \cdot (A\vec{x}) = \vec{x}^TA^TA\vec{x} = \vec{x}^T(A^TA\vec{x}) = \vec{x}^T\vec{0} = 0$$

It implies that the vector $A\vec{x} = \vec{0}$**. We conclude that**

$$\mathrm{Null}(A) = \mathrm{Null}(A^TA) \quad \Longrightarrow \quad \mathrm{rank}(A) = \mathrm{rank}(A^TA)$$

Thus if the matrix A **has linearly independent columns, then** $\mathrm{rank}(A) = n = \mathrm{rank}(A^TA)$ **and the matrix** A^TA **is invertible.**