

UNIVERSITY OF BERGEN
The Faculty of Mathematics and Natural Sciences

Exam in MAT121 - Linear algebra

September, 28, 2020, from 09.00 to 15.00

- Allowed help resources: all, except of communication between students

The exam consists of two parts:

The first set of exercises is of type “multiple choice”. You have to choose the correct answer and mark it. This part assumes that you give answers on the computer.

The second set of exercises requires from you an ability to make a proof of some statement. If you have difficulty to write it on the computer, just write it by hand on the additional ark and deliver.

1.1 Consider the matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}.$$

The set of all values of $\{a, b, c \in \mathbb{R}\}$ for which the matrix A is diagonalizable is given by: (choose the correct answer)

3 points

Solution to 1.1 Since the matrix A has three different eigenvalues, it is always diagonalisable.

- all possible $a, b, c \in \mathbb{R}$.

2.1 The set of all values of x for which the matrix

$$A = \begin{bmatrix} 2 & 0 & 10 \\ 0 & 7+x & -3 \\ 0 & 4 & x \end{bmatrix}$$

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is not invertible is given by: (choose the correct answer)

3 points

Solution to 2.1 We have

$$\det(A) = x^2 + 7x + 12 = (x + 3)(x + 4) = 0$$

for $x = -3$, $x = -4$.

- $x = -3$, $x = -4$

3.1 Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of some (2×2) matrix A corresponding to the eigenvalue 3 and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue -2 .

Then $A^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is equal to: (choose the correct answer)

4 points

Solution to 3.1 Since

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

we have

$$A^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = A \left(A \left(2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \right) = A \left(6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 26 \\ 22 \end{bmatrix}.$$

- $\begin{bmatrix} 26 \\ 22 \end{bmatrix}$

4.1 Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

be a basis of \mathbb{R}^3 . Then the vector

$$\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has the following coordinates in the basis \mathcal{B} : (choose the correct answer)

4 points

Solution to 4.1 We need to solve the system

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ b+c \\ c \end{bmatrix}$$

•

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a+b+c \\ b+c \\ c \end{bmatrix}$$

5.1 Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

defines the linear transformation from \mathbb{R}^4 to \mathbb{R}^5 by the rule $T(\vec{x}) = A\vec{x}$. Denote \mathcal{B}_K the basis of $\text{Ker}(T)$, \mathcal{B}_R the basis of range of T , and \mathcal{B}_{Rw} the basis of $\text{Row}(A)$. Then the bases are given by the following sets: (choose the correct answer)

4 points

Solution to 5.1 We have

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It gives the answer

•

$$\mathcal{B}_K = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{B}_R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\},$$

$$\mathcal{B}_{Rw} = \{ [1, -1, 0, 0], [0, 1, 1, 1] \}.$$

6.1 Let A be a square matrix that has the characteristic polynomial

$$\det(A - \lambda I) = (\lambda - 1)^2(\lambda - 2)^3(\lambda - 3)^4(\lambda - 4)^5.$$

The rank of the matrix A is equal to: (choose the correct answer)

4 points

Solution to 6.1 The size of the matrix A is the number of eigenvalues counting with multiplicity. Since the zero is not the eigenvalue, then dimension of the null space is 0. It gives

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7.1 Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T(\vec{x}) = \text{proj}_{\text{span}\{\vec{u}\}}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}.$$

The standard matrix A_T of the transformation T and the rank r of T are given by: (choose the correct answer)

3 points

Solution to 7.1 We have

$$T(\vec{e}_1) = \text{proj}_{\text{span}\{\vec{u}\}}(\vec{e}_1) = \frac{1}{2} \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = \text{proj}_{\text{span}\{\vec{u}\}}(\vec{e}_2) = \frac{1}{2} \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Since the image of T belongs to one dimensional space $L = \text{span}\{\vec{u}\}$ the rank of map T is equal to 1.

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$$A_T = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad r = 1$$

8.1 Let $\mathbb{M}_{2 \times 2}$ be the space of (2×2) matrices. Let

$$V = \left\{ A \in \mathbb{M}_{2 \times 2} \mid A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, a, b, c \in \mathbb{R} \right\}.$$

We denote \mathbb{B} the basis of V and $d = \dim(V)$. Then the basis and the dimension of V are given by: (choose the correct answer)

3 points

Solution to 8.1 Since any matrix from V can be written as

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It gives the answer

•

$$\mathbb{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}, \quad d = 3$$

9.1 Let \mathbb{P}_2 be the vector space of all polynomials of degree 2 or less. Consider the set of polynomials

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$\begin{aligned} p_1(t) &= -1 + t - 2t^2, & p_2(t) &= t + 3t^2, \\ p_3(t) &= 1 + 2t + 8t^2, & p_4(t) &= 1 + t + t^2. \end{aligned}$$

The following polynomials from the set S form the basis \mathcal{B} of \mathbb{P}_2 : (choose the correct answer)

3 points

Solution to 9.1 We have the following matrix of coordinates of polynomials in the standard basis $\{1, t, t^2\}$:

$$\begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ -2 & 3 & 8 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -3 & -7 \end{bmatrix}$$

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$$\mathcal{B} = \{p_1(t), p_2(t), p_3(t)\}$$

10.1 Let \mathbb{P}_2 be the vector space of all polynomials of degree 2 or less. Consider the set of polynomials

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$\begin{aligned} p_1(t) &= -1 + t - 2t^2, & p_2(t) &= t + 3t^2, \\ p_3(t) &= 1 + 2t + 8t^2, & p_4(t) &= 1 + t + t^2. \end{aligned}$$

The coordinates of the given polynomials in the basis \mathcal{B} chosen from the set S are given by: (choose the correct answer)

3 points

Solution to 10.1 We have the following matrix of coordinates of polynomials in the standard basis $\{1, t, t^1\}$:

$$\begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ -2 & 3 & 8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & \frac{7}{3} \end{bmatrix}$$

•

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix} -\frac{4}{3} \\ 5 \\ -\frac{7}{3} \end{bmatrix}$$

11.1 Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 \\ x_1 + x_2 \end{bmatrix}.$$

The orthonormal basis of the range of T is given by

3 points

Solution to 11.1 The standard matrix A_T for transformation is

$$A_T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

•

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

12.1 Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 3 & 3 & 0 \end{bmatrix}.$$

Then

$$\det \left(A^2 (C^3 B)^{-1} A^{-2} (B^T)^2 C^4 \right)$$

equal to: (choose the correct answer)

4 points

Solution to 12.1 We have

$$A^2 (C^3 B)^{-1} A^{-2} (B^T)^2 C^4 = A^2 B^{-1} C^{-3} A^{-2} (B^T)^2 C^4$$

By the properties $\det(PS) = \det(P)\det(S)$, $\det(P) = \det(P^T)$ we obtain

$$\begin{aligned} \det \left(A^2 B^{-1} C^{-3} A^{-2} (B^T)^2 C^4 \right) &= \det \left(A^2 A^{-2} B^{-1} B^2 C^{-3} C^4 \right) \\ &= \det(BC) = 24 \cdot (-24). \end{aligned}$$

- -24^2

13.1 The quadratic form

$$Q = 4x^2 + 4xy + 4y^2 + 4xz + 4z^2 + 4yz$$

is: (choose the correct answer)

3 points

Solution to 13.1 The matrix of the quadratic form has eigen values

$$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \lambda_3 = 8.$$

- positive definite

14.1 Let

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The value c for which the vector

$$\frac{1}{2}\vec{u} - c\vec{v}$$

is unit: (choose the correct answer)

3 points

Solution to 14.1 We have to find the value of c that satisfies the equation

$$\left(\frac{1}{2}\vec{u} - c\vec{v}\right) \cdot \left(\frac{1}{2}\vec{u} - c\vec{v}\right) = 10c^2 + 5c + \frac{3}{2} = 1$$

The quadratic equation

$$20c^2 + 10c + 1$$

has roots

- $c = \frac{-5 \pm \sqrt{5}}{20}$

15.1 Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$. The spectral decomposition of A is given by: (choose the correct answer)

3 points

Solution to 15.1 The eigenvalues and orthonormal eigenvectors of A are

$$\lambda_1 = 1, \quad \vec{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$$\lambda_2 = 11, \quad \vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then

$$A = \lambda_1 \vec{u}_1 (\vec{u}_1)^T + \lambda_2 \vec{u}_2 (\vec{u}_2)^T =$$

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$$A = \frac{1}{5} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + \frac{11}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

16.1. The distance from $\vec{b} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix} \in \mathbb{R}^3$ to a subspace

$$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 - x_3 = 0\}$$

is equal to: (choose the correct answer) (choose the correct answer)

3 points

Solution to 16.1 The vector $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is the normal vector to W . Then the distance is equal to

$$\text{dist}(\vec{b}, W) = \left\| \text{proj}_{\text{span}\{\vec{n}\}}(\vec{b}) \right\| = \left\| \frac{-2}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\| =$$

- $\frac{2}{\sqrt{3}}$

17.1 The least square line $y = \beta_0 + \beta_1 x$ that best fits the data points

$$\{(2, 3), (3, 2), (5, 1), (6, 0)\}$$

is given by: (choose the correct answer)

4 points

Solution to 17.1 We have to find the least square solution to the equation

$$X\vec{\beta} = \vec{y},$$

where

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

The solution is given by

$$\beta = (X^T X)^{-1} X^T \vec{y},$$

where

$$X^T X = \begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix}, \quad (X^T X)^{-1} = \frac{1}{20} \begin{bmatrix} 37 & -8 \\ -8 & 2 \end{bmatrix}, \quad X^T \vec{y} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

- $y = \frac{43}{10} - \frac{7}{10}x$

18.1 The following equation

$$5x_1^2 - 4x_1x_2 + 5x_2^2 = 122$$

defines: (choose the correct answer)

3 points

Solution to 18.1 The eigenvalues of the matrix

$$\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

of the quadratic form are $\lambda_1 = 3$, $\lambda_2 = 7$.

- an ellipse

19.1 Let

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}.$$

The orthogonal basis of $W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is given by: (choose the correct answer)

4 points

Solution to 19.1 Use the Gram-Schmidt procedure

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$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

20.1. Let

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$

The matrix R in the QR factorisation of A is given by: (choose the correct answer)

4 points

Solution to 20.1 Orthogonalizing the columns in A we obtain

$$Q = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Then

$$R = Q^T A$$

-

$$R = \frac{1}{2\sqrt{3}} \begin{bmatrix} 12 & -36 & 6 \\ 0 & 12 & 30 \\ 0 & 0 & 12 \end{bmatrix}$$

21.1. Let

$$\mathcal{A} = \left\{ \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\}, \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}.$$

Then the matrix $\mathcal{P}_{\mathcal{A} \rightarrow \mathcal{B}}$ of change of coordinates from the basis \mathcal{A} to the basis \mathcal{B} and the matrix $\mathcal{P}_{\mathcal{B} \rightarrow \mathcal{A}}$ of change of coordinates from the basis \mathcal{B} to the basis \mathcal{A} are given by

4 points

Solution to 21.1 We have

$$\begin{aligned} \mathcal{P}_{\mathcal{A} \rightarrow \mathcal{E}} &= \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}, & \mathcal{P}_{\mathcal{B} \rightarrow \mathcal{E}} &= \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} \\ \mathcal{P}_{\mathcal{E} \rightarrow \mathcal{A}} &= \frac{1}{8} \begin{bmatrix} -1 & 3 \\ -5 & 7 \end{bmatrix}, & \mathcal{P}_{\mathcal{E} \rightarrow \mathcal{B}} &= -\frac{1}{8} \begin{bmatrix} 2 & 2 \\ 5 & 1 \end{bmatrix} \end{aligned}$$

Then

$$\mathcal{P}_{\mathcal{A} \rightarrow \mathcal{B}} = \mathcal{P}_{\mathcal{E} \rightarrow \mathcal{B}} \mathcal{P}_{\mathcal{A} \rightarrow \mathcal{E}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}, \quad \mathcal{P}_{\mathcal{B} \rightarrow \mathcal{A}} = \mathcal{P}_{\mathcal{A} \rightarrow \mathcal{B}}^{-1}$$

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$$\mathcal{P}_{\mathcal{A} \rightarrow \mathcal{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}, \quad \mathcal{P}_{\mathcal{B} \rightarrow \mathcal{A}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

22.1. Let

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

be a basis in \mathbb{R}^2 and $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear transformation given by $\vec{x} \mapsto A\vec{x}$, where

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}.$$

Then the \mathcal{B} -matrix $M_{\mathcal{B}}$ for the linear transformation T in the basis \mathcal{B} is given by: (choose the correct answer)

4 points

Solution to 22.1 We have

$$\begin{aligned} \mathcal{P}_{\mathcal{B} \rightarrow \mathcal{E}} &= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, & \mathcal{P}_{\mathcal{E} \rightarrow \mathcal{B}} &= \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}, \\ M_{\mathcal{B}} &= \mathcal{P}_{\mathcal{E} \rightarrow \mathcal{B}} A \mathcal{P}_{\mathcal{B} \rightarrow \mathcal{E}} = \end{aligned}$$

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$$M_B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

23.1. Let A be a (6×4) matrix and B a (4×6) matrix. Show that 6×6 matrix AB can not be invertible.

8 points

Solution to 23.1 If the matrix AB would be invertible, then the rank of AB would be 6, or in other words AB would have 6 linearly independent columns, but this is impossible by the following. By the definition of the product of matrices we have

$$AB = [A\vec{b}_1 \ \dots \ A\vec{b}_6].$$

The (4×6) matrix B has maximum 4 linearly independent columns. So, at least two of columns are linear combination of other four of them. Assume that

$$\vec{b}_5 = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 + c_4 \vec{b}_4,$$

but then

$$A\vec{b}_5 = A(c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 + c_4 \vec{b}_4) = c_1 A\vec{b}_1 + c_2 A\vec{b}_2 + c_3 A\vec{b}_3 + c_4 A\vec{b}_4.$$

Thus in the matrix AB the fifth column $A\vec{b}_5$ will be also linearly dependent of four first columns in AB and therefore AB can not have 6 linearly independent columns.

24.1. Let A be $(n \times n)$ real matrix and suppose that A can be written in the form

$$A = URU^T,$$

where U is an orthogonal matrix and R is an upper triangular matrix. Show that A and R have the same characteristic polynomial. Deduce that A has n real eigenvalues, counting multiplicity.

8 points

Solution to 24.1 Since U is orthogonal we have that $UU^T = I$. We calculate

$$\begin{aligned} \det(A - \lambda I) &= \det(URU^T - \lambda I) = \det(URU^T - \lambda UU^T) \\ &= \det(U(R - \lambda I)U^T) = \det U \det(R - \lambda I) \det(U^T) \\ &= \det(R - \lambda I) \det(UU^T) = \det(R - \lambda I) \det(I) \\ &= \det(R - \lambda I) \end{aligned}$$

that shows that A and R has the same characteristic polynomials. Since the matrix R is a real upper triangular, the eigenvalues of R equals to diagonal elements and

coincide with the eigenvalues of A , that shows that all the eigenvalues of A are real and it is n of them.

25.1. Suppose that

$$\{\vec{v}_1, \vec{v}_2\}$$

are linear independent in 5-dimensional vector space V . Prove that

$$\{\vec{u}_1 = \vec{v}_1 + 2\vec{v}_2, \vec{u}_2 = 2\vec{v}_1 - \vec{v}_2\}$$

are also linear independent in V .

8 points

Solution to 25.1 Since $\{\vec{v}_1, \vec{v}_2\}$ are linearly independent then the equality $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ implies $c_1 = c_2 = 0$. We calculate

$$d_1\vec{u}_1 + d_2\vec{u}_2 = d_1(\vec{v}_1 + 2\vec{v}_2) + d_2(2\vec{v}_1 - \vec{v}_2) = (d_1 + 2d_2)\vec{v}_1 + (2d_1 - d_2)\vec{v}_2.$$

If $d_1\vec{u}_1 + d_2\vec{u}_2 = \vec{0}$ then also

$$(d_1 + 2d_2)\vec{v}_1 + (2d_1 - d_2)\vec{v}_2 = \vec{0}.$$

The latter equality implies that

$$\begin{cases} d_1 + 2d_2 = 0 \\ 2d_1 - d_2 = 0 \end{cases} \implies \begin{cases} 5d_1 = 0 \\ 2d_1 = d_2 \end{cases} \implies \begin{cases} d_1 = 0 \\ d_2 = 0 \end{cases}.$$

Thus we showed that the equality $d_1\vec{u}_1 + d_2\vec{u}_2 = \vec{0}$ implies $d_1 = d_2 = 0$ that proves that the vectors $\{\vec{u}_1, \vec{u}_2\}$ are linearly independent.