English

UNIVERSITY OF BERGEN

The Faculty of Mathematics and Natural Sciences

Exam in MAT121 - Linear algebra

June, 06, 2020, from 09.00 to 15.00

- Allowed help resources: all, except of communication between students

The exam consists of two parts:

The first set of exercises is of type "multiple choice". You have to choose the correct answer and mark it. This part assumes that you give answers on the computer.

The second set of exercises requires from you an ability to make a proof of some statement. If you have difficulty to write it on the computer, just write it by hand on the additional ark and deliver.

1.1 Consider the vectors:

$$\overrightarrow{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \overrightarrow{x}_2 = \begin{bmatrix} h \\ 1 \\ -h \end{bmatrix}, \qquad \overrightarrow{x}_3 = \begin{bmatrix} 1 \\ 2h \\ 3h+1 \end{bmatrix}.$$

The set of all values of h for which $\{\overrightarrow{x}_1, \overrightarrow{x}_2, \overrightarrow{x}_2\}$ are linearly independent is given by: (choose the correct answer)

- ○ $h \neq -1$, $h \neq -\frac{1}{2}$
- ○ $h = -1$, $h = -\frac{1}{2}$
- ○ $h = 0, 1, 2, \ldots$
- ○ for all values of $h \neq 0$
- ○ non of them

3 points

2.1 The matrix
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$
is not invertible if: (choose the correct answer)

- ○ $k = 1,\ k = 2$
- ○ $k = 0$
- ○ $k \neq 1,\ k \neq 2$
- ○ $k^2 - k = 0$
- ○ non of them

3 points

3.1 Suppose the following information is known about a (3×3) matrix A:
$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$
Then the matrix A has the following eigenvalues: (choose the correct answer)

- ○ $\lambda_1 = 6, \quad \lambda_2 = 3, \quad \lambda_3 = 0$
- ○ $\lambda_1 = 6, \quad \lambda_2 = 3, \quad \lambda_3 = 3$
- ○ $\lambda_1 = 6, \quad \lambda_2 = 3$
- ○ $\lambda_1 = 6, \quad \lambda_2 = 3$, and λ_3 is impossible to find
- ○ non of them

4 points

4.1 Suppose the following information is known about a (3×3) matrix A:
$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$
Then the following vectors are eigenvectors of A: (choose the correct answer)

- ○ $\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$

- ○ $\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\begin{bmatrix}0\\0\\0\end{bmatrix}$

- ○ $\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\1\end{bmatrix}$

- ○ $\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\1\end{bmatrix}$, and the eigenvector $\overrightarrow{v}_3$ is impossible to find

- ○ non of them

4 points

5.1 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & -3 & 4 & 5 \\ -3 & 10 & -6 & -7 \end{bmatrix}.$$

We denote $\mathcal{B}_C$ the basis of the column space Col (A), $\mathcal{B}_R$ the basis of the row space Row (A), and $\mathcal{B}_{ON}$ an orthonormal basis of the null space Null (A) of A. The mentioned above bases are given by: (choose the correct answer)

- ○

$$\mathcal{B}_C = \left\{ \begin{bmatrix}0\\1\\-3\end{bmatrix}, \begin{bmatrix}1\\-3\\10\end{bmatrix}, \begin{bmatrix}2\\4\\-6\end{bmatrix} \right\},$$

$$\mathcal{B}_R = \left\{ \begin{bmatrix}1, & 0, & 0, & \frac{3}{2}\end{bmatrix}, \begin{bmatrix}0, & 1, & 0, & \frac{1}{2}\end{bmatrix}, \begin{bmatrix}0, & 0, & 1, & \frac{5}{4}\end{bmatrix} \right\},$$

$$\mathcal{B}_{ON} = \left\{ \frac{1}{9} \begin{bmatrix} -6 \\ -2 \\ -5 \\ 4 \end{bmatrix}, \right\},$$

- $$\mathcal{B}_C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \right\},$$

$$\mathcal{B}_R = \left\{ \begin{bmatrix} 1, & -3, & 4, & 5 \end{bmatrix}, \begin{bmatrix} 0, & 1, & 2, & 3 \end{bmatrix}, \begin{bmatrix} 0, & 0, & 4, & 5 \end{bmatrix} \right\},$$

$$\mathcal{B}_{ON} = \left\{ \frac{1}{\sqrt{59}} \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}, \right\},$$

- $$\mathcal{B}_C = \left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right\},$$

$$\mathcal{B}_R = \left\{ \begin{bmatrix} 1, & 0, & 0, & \frac{3}{2} \end{bmatrix}, \begin{bmatrix} 0, & 1, & 0, & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0, & 0, & 1, & \frac{5}{4} \end{bmatrix} \right\},$$

$$\mathcal{B}_{ON} = \left\{ \frac{4}{5\sqrt{13}} \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{5}{4} \end{bmatrix}, \right\},$$

○

$$\mathcal{B}_C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

$$\mathcal{B}_R = \left\{ \begin{bmatrix} 1, & 0, & 0, & \frac{3}{2} \end{bmatrix}, \begin{bmatrix} 0, & 1, & 0, & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0, & 0, & 1, & \frac{5}{4} \end{bmatrix} \right\},$$

$$\mathcal{B}_{ON} = \left\{ \frac{1}{9} \begin{bmatrix} -6 \\ -2 \\ -5 \\ 4 \end{bmatrix}, \right\}$$

○ non of them

4 points

6.1 Suppose that the matrix A is diagonalizable and has the characteristic polynomial

$$\det(A - I(\lambda)) = \lambda^2(\lambda - 3)(\lambda + 2)^3(\lambda - 4)^3.$$

Let $(m \times n)$ be the size of the matrix A, d is the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 4$ and $p = \dim\big(\text{Null}\,(A)\big)$. Which of the following numbers correspond to the matrix A? (choose the correct answer)

○

$$m \times n = 9 \times 9, \quad d = 3, \quad p = 2$$

○

$$m \times n = 9 \times 4, \quad d = 3, \quad p = 2$$

○

$$m \times n = 4 \times 4, \quad d = 1, \quad p = 0$$

○

$$m \times n = 9 \times 9, \quad d = 3, \quad p = 0$$

○ non of them

4 points

7.1 Let A, B, C be $(n \times n)$ invertible matrices. When you simplify the expression

$$C^{-1}(AB^{-1})^{-1}(CA^{-1})^{-1}C^2$$

which matrix do you get? (choose the correct answer)

○ $C^{-1}BC$

○ A

○ $C^{-1}A^{-1}BC^{-1}AC^2$

○ B

○ non of them

3 points

8.1 Let $\mathbb{P}_3$ be the vector space of all polynomials of degree 3 or less. Let

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\}, \qquad Q = \text{span}\{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$p_1(t) = 1 + 3t + 2t^2 - t^3, \quad p_2(t) = t + t^3,$$

$$p_3(t) = t + t^2 - t^3, \quad p_4(t) = 3 + 8t + 8t^3.$$

The coordinates of the polynomials $\{p_1(t), p_2(t), p_3(t), p_4(t)\}$ in the standard basis $\mathcal{E} = \{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ are: (choose the correct answer)

○

$$[p_1(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 8 \\ 0 \\ 8 \end{bmatrix}$$

○

$$[p_1(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

○

$$[p_1(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

○

$$[p_1(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 8 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 8 \end{bmatrix}$$

○ non of them

3 points

9.1 Let $\mathbb{P}_3$ be the vector space of all polynomials of degree 3 or less. Let

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\}, \qquad Q = \text{span}\{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$p_1(t) = 1 + 3t + 2t^2 - t^3, \quad p_2(t) = t + t^3,$$

$$p_3(t) = t + t^2 - t^3, \quad p_4(t) = 3 + 8t + 8t^3.$$

The basis $\mathcal{B}$ of Q chosen from the set S is given by: (choose the correct answer)

○

$$p_1(t), \quad p_2(t), \quad p_3(t)$$

○

$$p_1(t), \quad p_2(t), \quad p_3(t), \quad p_4(t)$$

○

$$p_1(t), \quad p_3(t), \quad p_4(t)$$

○

$$p_2(t), \quad p_3(t), \quad p_4(t)$$

○ non of them

3 points

10.1 Let $\mathbb{P}_3$ be the vector space of all polynomials of degree 3 or less. Let

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\}, \qquad Q = \text{span}\{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$p_1(t) = 1 + 3t + 2t^2 - t^3, \quad p_2(t) = t + t^3,$$

$$p_3(t) = t + t^2 - t^3, \quad p_4(t) = 3 + 8t + 8t^3.$$

The polynomials in S has the following coordinates in the basis $\mathcal{B}$ of Q: (choose the correct answer). Remember that the basis $\mathcal{B}$ is chosen from the polynomial belonging to the set S.

○

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}$$

○

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

○

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 8 \\ 0 \\ 8 \end{bmatrix}$$

○

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}$$

○ non of them

4 points

11.1 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}. \tag{1}$$

The standard matrix A of T, rank r of T, and $d = \dim(\ker(T))$ are given by: (choose the correct answer)

○

$$A = \begin{bmatrix} 3 & -4 \\ 16 & -23 \\ 7 & -9 \end{bmatrix}, \quad r = 2, \quad d = 0.$$

○

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \quad r = 2, \quad d = 1.$$

○

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -5 \\ 3 & 1 \end{bmatrix}, \quad r = 3, \quad d = 0.$$

○

$$A = \begin{bmatrix} 3 & -4 \\ 16 & -23 \\ 7 & -9 \end{bmatrix}, \quad r = 3, \quad d = 0.$$

○ non of them

4 points

12.1 Let

$$A = \begin{bmatrix} 1 & -14 & 4 \\ -1 & 6 & -2 \\ -2 & 24 & -7 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 4 \\ -1 \\ -7 \end{bmatrix}.$$

Then $A^{10}\vec{v}$ is equal to the vector: (choose the correct answer).

You can use the following information without proving it: the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 1$, and the corresponding eigenspaces are

$$\text{span}\left\{\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}\right\}, \quad \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right\}, \quad \text{span}\left\{\begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}\right\}.$$

○

$$\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

○

$$\begin{bmatrix} 1 & 14^{10} & 4^{10} \\ 1 & 6^{10} & 2^{10} \\ 2^{10} & 24^{10} & 7^{10} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ -7 \end{bmatrix} = \begin{bmatrix} 4 - 14^{10} - 7 \cdot 4^{10} \\ 4 - 6^{10} - 7 \cdot 2^{10} \\ 4 \cdot 2^{10} - 24^{10} - 7^{11} \end{bmatrix}$$

○

$$\begin{bmatrix} 40 \\ -18 \\ -69 \end{bmatrix}$$

○

$$\begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

○ non of them

4 points

13.1 The quadratic form

$$Q = x^2 - 4xz + 5y^2 + 4z^2$$

is: (choose the correct answer)

- ○ positive semidefinite
- ○ positive definite
- ○ negative semidefinite
- ○ negative definite
- ○ non of them

3 points

14.1 Let

$$\overrightarrow{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \qquad \overrightarrow{v} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The value c for which the vectors

$$\overrightarrow{x}_1 = c\overrightarrow{u} + \overrightarrow{v} \quad \text{and} \quad \overrightarrow{x}_2 = \overrightarrow{u} + c\overrightarrow{v}$$

are orthogonal is: (choose the correct answer)

- ○ $c = \frac{8 \pm \sqrt{39}}{5}$
- ○ $c = 0$
- ○ $c = \frac{8 \pm \sqrt{89}}{5}$
- ○ $c = -1$
- ○ non of them

3 points

15.1 Let

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

The spectral decomposition of A is given by: (choose the correct answer)

- ○

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- ○ $$A = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
- ○ $$A = 2\overrightarrow{u}_1^T\overrightarrow{u}_1 + 4\overrightarrow{u}_2^T\overrightarrow{u}_2$$
- ○ $$A = 2\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
- ○ non of them

3 points

16.1. The distance from $\overrightarrow{b} = \begin{bmatrix} 3 \\ 5 \\ -9 \end{bmatrix} \in \mathbb{R}^3$ to a subspace

$$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 + x_2 + x_3 = 0\}$$

is equal to: (choose the correct answer)

- ○ $\frac{1}{\sqrt{3}}$
- ○ $-\frac{1}{\sqrt{3}}$
- ○ 1
- ○ -1
- ○ non of them

4 points

17.1 A certain experiment produces the data

$$\{(0, 2), (-1, -1), (2, 0), (1, 1)\}.$$

The least square curve $y = \beta_1 x + \beta_2 x^2$ is given by: (choose the correct answer)

- ○ $y = \frac{9}{11}x - \frac{4}{11}x^2$
- ○ $y = -\frac{9}{11}x + \frac{4}{11}x^2$
- ○ $y = 36x - 8x^2$

- $y = -36x + 8x^2$

- non of them

3 points

18.1 The following equation

$$2x_1^2 - 6x_1x_2 - 6x_2^2 = \pi$$

defines: (choose the correct answer)

- a hyperbola

- an ellipse

- a plane

- a straight line

- non of them

3 points

19.1 Let

$$\vec{x}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}.$$

An orthogonal basis for $V = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is given by the following vectors: (choose the correct answer)

- $$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

- $$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -7 \\ 0 \\ -3 \\ 6 \end{bmatrix}$$

○

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

○

$$\vec{v}_1 = \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{86}} \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \vec{v}_3 = \frac{1}{\sqrt{70}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}$$

○ non of them

3 points

20.1. Let

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

The matrix R in the QR factorisation of A is given by: (choose the correct answer)

○

$$R = \frac{1}{\sqrt{5}} \begin{bmatrix} 10 & -20 & 15 \\ 0 & 10 & -5 \\ 0 & 0 & 10 \end{bmatrix}$$

○

$$R = \begin{bmatrix} 20 & -40 & 30 \\ 0 & 20 & -10 \\ 0 & 0 & 20 \end{bmatrix}$$

○

$$R = \sqrt{5} \begin{bmatrix} 10 & -20 & 15 \\ 0 & 10 & -5 \\ 0 & 0 & 10 \end{bmatrix}$$

○

$$R = \begin{bmatrix} 2 & -4 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

○ non of them

4 points

21.1. Let

$$\mathcal{A} = \{\overrightarrow{a}_1, \overrightarrow{a}_2, \overrightarrow{a}_3\}, \qquad \mathcal{B} = \left\{\overrightarrow{b}_1, \overrightarrow{b}_2, \overrightarrow{b}_3\right\},$$

be two bases of a vector space V. Suppose that

$$\overrightarrow{a}_1 = 4\overrightarrow{b}_1 - \overrightarrow{b}_2, \quad \overrightarrow{a}_2 = -\overrightarrow{b}_1 + \overrightarrow{b}_2 - \overrightarrow{b}_3, \quad \overrightarrow{a}_3 = \overrightarrow{b}_2 - 2b_3.$$

Let also

$$\overrightarrow{x} = 3\overrightarrow{a}_1 + 4\overrightarrow{a}_2 + \overrightarrow{a}_3.$$

Then the change-of-coordinates matrix $\mathcal{P}_{\mathcal{A}\to\mathcal{B}}$ from the basis $\mathcal{A}$ to the basis $\mathcal{B}$ and $[\overrightarrow{x}]_{\mathcal{B}}$ are given by: (choose the correct answer)

○

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

○ non of them

3 points

22.1. Let

$$\mathcal{B} = \left\{\overrightarrow{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ \overrightarrow{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

be a basis for $\mathbb{R}^2$ and $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by $\overrightarrow{x} \mapsto A\overrightarrow{x}$, where

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}.$$

Then the $\mathcal{B}$-matrix $M_{\mathcal{B}}$ for the linear transformation T with respect to the basis $\mathcal{B}$ is given by: (choose the correct answer)

- ○ $$M_{\mathcal{B}} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$
- ○ $$M_{\mathcal{B}} = \begin{bmatrix} 5 & 25 \\ 0 & 5 \end{bmatrix}$$
- ○ $$M_{\mathcal{B}} = \frac{1}{5}\begin{bmatrix} 17 & 9 \\ -16 & -7 \end{bmatrix}$$
- ○ $$M_{\mathcal{B}} = \begin{bmatrix} 17 & 9 \\ -16 & -7 \end{bmatrix}$$
- ○ non of them

4 points

23.1. Suppose that $\{\overrightarrow{v}_1, \overrightarrow{v}_2\}$ are linearly independent in 5-dimensional vector space V. Prove that

$$\{\overrightarrow{u}_1 = \overrightarrow{v}_1 + \overrightarrow{v}_2,\ \overrightarrow{u}_2 = \overrightarrow{v}_1 - \overrightarrow{v}_2\}$$

are also linearly independent in V.

8 points

24.1. Let A be a (5×3) matrix. Suppose that there is a (3×5) matrix B such that

$$BA = I_3.$$

Suppose further that the system $A\overrightarrow{x} = \overrightarrow{b}$ has at least one solution. Prove that this solution is actually a unique solution.

8 points

25.1. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The sum of the diagonal entries in A is called the trace of the matrix A and it is denoted by $\operatorname{tr} A = a + d$.

Show that the characteristic polynomial of A can be written as

$$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A.$$

Hence give the condition for A to have real eigenvalues.

8 points

Professor Irina Markina

Good luck!