English

University of Bergen

The Faculty of Mathematics and Natural Sciences

Exam in MAT121 - Linear algebra

June 08, 2021, from 09.00 to 14.30

- Allowed help resources: all, except of communication between students

The exam consists of two parts:

The first set of exercises is of type "multiple choice". You have to choose the correct answer and mark it. Some of the questions can have several correct answers. This part assumes that you give answers on the computer.

The second set of exercises requires from you an ability to make a proof of some statement. If you have difficulty to write it on the computer, just write it by hand on the additional ark and deliver.

Maximal points is 100.

1. Number of solutions

1. For which values of k the system has a unique solution

$$\begin{cases} x_1 + 2x_2 + kx_3 + 3x_4 = k \\ x_1 + 2x_2 + k^2x_3 + 3x_4 = 0 \end{cases}$$

- non of them
- ○ $k = 1$
- ○ $k \neq 0$
- ○ $k \neq 1$
- ○ $k = 0$

2. For which values of k the system has infinitely many solutions

$$\begin{cases} x_1 + 2x_2 + kx_3 + 3x_4 = k \\ x_1 + 2x_2 + k^2x_3 + 3x_4 = 0 \end{cases}$$

- ● $k = 0$
- ○ $k = 1$
- ○ $k \neq 0$
- ○ $k \neq 1$
- ○ non of them

3. For which values of k the system has no solutions

$$\begin{cases} x_1 + 2x_2 + kx_3 + 3x_4 = k \\ x_1 + 2x_2 + k^2x_3 + 3x_4 = 0 \end{cases}$$

- ● $k = 1$
- ○ $k = 0$
- ○ $k \neq 0$
- ○ $k \neq 1$
- ○ non of them

Solution

$$\begin{bmatrix} 1 & 2 & k & 3 & |k \\ 1 & 2 & k^2 & 3 & |0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & k & 3 & |k \\ 0 & 0 & k(k-1) & 0 & |-k \end{bmatrix}$$

$k = 0$ infinitely many
$k = 1$ no solutions
never unique solution

4. For which values of k the system has no solutions

$$\begin{cases} x_1 + 2x_2 = k - 2 \\ -x_1 - 2x_2 = -1 \\ x_1 + kx_2 = 0 \end{cases}$$

- ● $k = 2$
- ○ $k = 3$
- ○ $k \neq 2$
- ○ $k \neq 3$
- ○ non of them

5. For which values of k the system has a unique solution

$$\begin{cases} x_1 + 2x_2 = k - 2 \\ -x_1 - 2x_2 = -1 \\ x_1 + kx_2 = 0 \end{cases}$$

- ● $k = 3$
- ○ $k = 2$
- ○ $k \neq 2$
- ○ $k \neq 3$
- ○ non of them

6. For which values of k the system has infinitely many solutions

$$\begin{cases} x_1 + 2x_2 = k - 2 \\ -x_1 - 2x_2 = -1 \\ x_1 + kx_2 = 0 \end{cases}$$

- ● non of them

- ○ $k = 2$
- ○ $k \neq 2$
- ○ $k \neq 3$
- ○ $k = 3$

Solution

$$\begin{bmatrix} 1 & 2 & |k-2 \\ -1 & -2 & |-1 \\ 1 & k & |0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & |k-2 \\ 0 & 0 & |k-3 \\ 0 & k-2 & |2-k \end{bmatrix}$$

$k = 3$ unique solution
$k = 2$ no solutions
never infinitely many

2. Determinant

1. Let A, B, C be (3×3) matrices, such that
$$\det(A) = 2, \quad \det(B) = \pi, \quad \det(C) = 1.$$
What is the value of
$$\det(3AB^TA^{-1}C)$$

- ● 27π
- ○ 3π
- ○ -12π
- ○ $27\pi^T$
- ○ non of them

2. Let A, B, C be (3×3) matrices, such that
$$\det(A) = 2, \quad \det(B) = \pi, \quad \det(C) = 1.$$
What is the value of
$$\det(3A^TB^{-1}A^{-1}C)$$

- ● $\frac{27}{\pi}$
- ○ 9π
- ○ $\frac{3}{\pi}$
- ○ 27π
- ○ non of them

3. Let A, B, C be (3×3) matrices, such that
$$\det(A) = 2, \quad \det(B) = \pi, \quad \det(C) = 1.$$
What is the value of
$$\det(A^2B^TA^{-1}3C)$$

- ● 54π
- ○ 6π

- ○ 18π
- ○ 27π
- ○ non of them

4. Let A, B, C be (3×3) matrices, such that

$$\det(A) = 2, \quad \det(B) = \pi, \quad \det(C) = 1.$$

What is the value of

$$\det(3A^T B^T A^{-1} C)$$

- ● non of them
- ○ 3π
- ○ 9π
- ○ $27\pi^T$
- ○ -3π

5. Let A, B, C be (3×3) matrices, such that

$$\det(A) = 2, \quad \det(B) = \pi, \quad \det(C) = 1.$$

What is the value of

$$\det(4(AB)^T A^{-1} C)$$

- ● 64π
- ○ -16π
- ○ 4π
- ○ 12π
- ○ non of them

6. Let A, B, C be (3×3) matrices, such that

$$\det(A) = 2, \quad \det(B) = \pi, \quad \det(C) = 1.$$

What is the value of

$$\det(4(AB)^{-1} AC)$$

- ● $\frac{64}{\pi}$
- ○ $\frac{4}{\pi}$
- ○ -4π

- $-\frac{64}{\pi}$
- non of them

3. Eigenvectors

1. Let us assume that the matrix

$$A = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 2$. Then the eigenvectors of the matrix A are equal to:

- ●
$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- ○ non of them

Solution Eigenvalue is 1

2. Let us assume that the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 2 \\ -2 & 0 & 0 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 0$. Then the eigenvectors of the matrix A are equal to:

- ●

$$\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- ○

$$\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- ○

$$\begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- ○

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- ○ non of them

Solution Eigenvalues are 3 and 0

3. Let us assume that the matrix

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 5 & -3 \\ 2 & 2 & 1 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$. Then the eigenvectors of the matrix A are equal to:

- ●

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- ○

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- ○ $$\begin{bmatrix}3\\1\\2\end{bmatrix},\quad \begin{bmatrix}0\\-1\\0\end{bmatrix},\quad \begin{bmatrix}0\\0\\0\end{bmatrix}$$
- ○ $$\begin{bmatrix}1\\2\\3\end{bmatrix},\quad \begin{bmatrix}0\\1\\1\end{bmatrix}$$
- ○ non of them

Solution Eigenvalues are 3 and 5.

4. Let us assume that the matrix

$$A = \begin{bmatrix}5 & 4 & -4\\4 & 5 & -4\\4 & 4 & -3\end{bmatrix}$$

has the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$. Then the eigenvectors of the matrix A are equal to:

- ● $$\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \begin{bmatrix}1\\0\\1\end{bmatrix},\quad \begin{bmatrix}-1\\1\\0\end{bmatrix}$$
- ○ $$\begin{bmatrix}-1\\-1\\-1\end{bmatrix},\quad \begin{bmatrix}-1\\1\\0\end{bmatrix},\quad \begin{bmatrix}0\\1\\0\end{bmatrix}$$
- ○ $$\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \begin{bmatrix}0\\-1\\0\end{bmatrix},\quad \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$$
- ○ $$\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \begin{bmatrix}-1\\1\\0\end{bmatrix}$$
- ○ non of them

Solution Eigenvalues are 1 and 5.

5. Let us assume that the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 2 & 0 \\ 0 & -2 & 4 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 2$. Then the eigenvectors of the matrix A are equal to:

- ● non of them

- ○
$$\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution Eigenvalues are 2 and 4.

6. Let us assume that the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

has an eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$. Then the eigenvectors of the matrix A are equal to:

- ● $$\begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}-1\\1\\0\end{bmatrix}$$
- ○ $$\begin{bmatrix}-1\\1\\-1\end{bmatrix}, \quad \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}$$
- ○ $$\begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\-1\\0\end{bmatrix}, \quad \begin{bmatrix}-1\\0\\1\end{bmatrix}$$
- ○ $$\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}-1\\1\\0\end{bmatrix}$$
- ○ non of them

Solution Eigenvalues are 3 and 1.

4. Eigenvalues

1. Suppose the following information is known about a (3×3) matrix A:

$$A\begin{bmatrix}1\\2\\1\end{bmatrix} = 5\begin{bmatrix}1\\2\\1\end{bmatrix}, \quad A\begin{bmatrix}1\\-1\\1\end{bmatrix} = 2\begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad A\begin{bmatrix}2\\-1\\0\end{bmatrix} = 2\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

Then the matrix A has the following eigenvalues and eigenvectors: (choose the correct answer)

- ●
$$\lambda_1 = 5, \overrightarrow{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

- ○
$$\lambda_1 = 5, \overrightarrow{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \lambda_3 = 2, \overrightarrow{v}_3 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

- ○
$$\lambda_1 = 5, \overrightarrow{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

- ○
$$\lambda_1 = 5, \overrightarrow{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix} \text{ and } (\lambda_3, \overrightarrow{v}_3) \text{ is impossible to find}$$

- ○ non of them

2. Suppose the following information is known about a (3×3) matrix A:

$$A\begin{bmatrix}1\\1\\-1\end{bmatrix} = 3\begin{bmatrix}1\\1\\-1\end{bmatrix}, \quad A\begin{bmatrix}1\\-1\\1\end{bmatrix} = 2\begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad A\begin{bmatrix}2\\-1\\0\end{bmatrix} = 3\begin{bmatrix}1\\1\\-1\end{bmatrix}.$$

Then the matrix A has the following eigenvalues and eigenvectors: (choose the correct answer)

- $$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

- ○ $$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 3, \overrightarrow{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

- ○ $$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- ○ $$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ and } (\lambda_3, \overrightarrow{v}_3) \text{ is impossible to find}$$

- ○ non of them

3. Suppose the following information is known about a (3×3) matrix A:

$$A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \overrightarrow{0}, \quad A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Then the matrix A has the following eigenvalues and eigenvectors: (choose the correct answer)

- $$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 0, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 0, \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 0, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and } (\lambda_2, \overrightarrow{v}_2) \text{ is impossible to find}$$

○ non of them

4. Suppose the following information is known about a (3×3) matrix A:

$$A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Then the matrix A has the following eigenvalues and eigenvectors: (choose the correct answer)

● non of them

○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 3, \overrightarrow{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- ○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ and } (\lambda_3, \overrightarrow{v}_3) \text{ is impossible to find}$$

- ○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

5. Suppose the following information is known about a (3×3) matrix A:

$$A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Then the matrix A has the following eigenvalues and eigenvectors: (choose the correct answer)

- ●

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

- ○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

- ○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- ○

$$\lambda_1 = 3, \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 2, \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \text{ and } (\lambda_3, \overrightarrow{v}_3) \text{ is impossible to find}$$

○ non of them

6. Suppose the following information is known about a (3×3) matrix A:

$$A\begin{bmatrix}1\\1\\-1\end{bmatrix} = 5\begin{bmatrix}1\\1\\-1\end{bmatrix}, \quad A\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad A\begin{bmatrix}2\\-1\\0\end{bmatrix} = 5\begin{bmatrix}1\\1\\-1\end{bmatrix}.$$

Then the matrix A has the following eigenvalues and eigenvectors: (choose the correct answer)

● $$\lambda_1 = 5, \overrightarrow{v}_1 = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \quad \lambda_2 = 0, \overrightarrow{v}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

○ $$\lambda_1 = 5, \overrightarrow{v}_1 = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \quad \lambda_2 = 0, \overrightarrow{v}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \lambda_3 = 5, \overrightarrow{v}_3 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

○ $$\lambda_1 = 5, \overrightarrow{v}_1 = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \quad \lambda_2 = 1, \overrightarrow{v}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

○ $$\lambda_1 = 5, \overrightarrow{v}_1 = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \quad \lambda_2 = 1, \overrightarrow{v}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \lambda_3 = 0, \overrightarrow{v}_3 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

○ non of them

5. Determinant

1. The determinant of the matrix

$$A = \begin{bmatrix} 3\cos(\frac{\pi}{48}) & \cos(\frac{\pi}{48}) & 5\cos(\frac{\pi}{48}) \\ 1 & 0 & 2 \\ 7\log 108 & \log 108 & 8\log 108 \end{bmatrix}$$

is equal to: (choose the correct answer)

- ● $5\cos(\frac{\pi}{48})\log 108$
- ○ $-5\cos(\frac{\pi}{48})\log 108$
- ○ $5\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ $-5\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ non of them

2. The determinant of the matrix

$$A = \begin{bmatrix} 3\cos(\frac{\pi}{48}) & \cos(\frac{\pi}{48}) & 5\cos(\frac{\pi}{48}) \\ -2 & 0 & -4 \\ 7\log 108 & \log 108 & 8\log 108 \end{bmatrix}$$

is equal to: (choose the correct answer)

- ● $-10\cos(\frac{\pi}{48})\log 108$
- ○ $10\cos(\frac{\pi}{48})\log 108$
- ○ $10\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ $-10\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ non of them

3. The determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3\cos(\frac{\pi}{48}) & \cos(\frac{\pi}{48}) & 5\cos(\frac{\pi}{48}) \\ 7\log 108 & \log 108 & 8\log 108 \end{bmatrix}$$

is equal to: (choose the correct answer)

- • $-5\cos(\frac{\pi}{48})\log 108$
- ○ $5\cos(\frac{\pi}{48})\log 108$
- ○ $5\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ $-5\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ non of them

4. The determinant of the matrix

$$A = \begin{bmatrix} 3\cos(\frac{\pi}{48}) & 1 & 5\log 108 \\ \cos(\frac{\pi}{48}) & 0 & 2\log 108 \\ 7\cos(\frac{\pi}{48}) & 1 & 8\log 108 \end{bmatrix}$$

is equal to: (choose the correct answer)

- • $5\cos(\frac{\pi}{48})\log 108$
- ○ $-5\cos(\frac{\pi}{48})\log 108$
- ○ $5\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ $-5\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ non of them

5. The determinant of the matrix

$$A = \begin{bmatrix} 3\cos(\frac{\pi}{48}) & 3 & 5\log 108 \\ \cos(\frac{\pi}{48}) & 0 & 2\log 108 \\ 7\cos(\frac{\pi}{48}) & 3 & 8\log 108 \end{bmatrix}$$

is equal to: (choose the correct answer)

- • $15\cos(\frac{\pi}{48})\log 108$

- ○ $-15\cos(\frac{\pi}{48})\log 108$
- ○ $15\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ $-15\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ non of them

6. The determinant of the matrix

$$A = \begin{bmatrix} 3\cos(\frac{\pi}{48}) & 5\log 108 & 2 \\ \cos(\frac{\pi}{48}) & 2\log 108 & 0 \\ 7\cos(\frac{\pi}{48}) & 8\log 108 & 2 \end{bmatrix}$$

is equal to: (choose the correct answer)

- ● non of them
- ○ $10\cos(\frac{\pi}{48})\log 108$
- ○ $-10\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ $10\Big(\cos(\frac{\pi}{48})\log 108\Big)^3$
- ○ $-5\cos(\frac{\pi}{48})\log 108$

6. Orthonormality

1. Which of the following systems of vectors is orthonormal: (choose the correct answer)

- ● $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \frac{1}{\sqrt{22}} \begin{bmatrix} -\sqrt{2} \\ -3 \\ \sqrt{2} \\ 3 \end{bmatrix}$

- ○ $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \frac{1}{22} \begin{bmatrix} -\sqrt{2} \\ 3 \\ \sqrt{2} \\ -3 \end{bmatrix}$

- ○ $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \frac{1}{\sqrt{22}} \begin{bmatrix} -\sqrt{2} \\ 3 \\ \sqrt{2} \\ 3 \end{bmatrix}$

- ○ $\begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- ○ non of them

2. Which of the following systems of vectors is orthonormal: (choose the correct answer)

- ● $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \frac{1}{\sqrt{22}} \begin{bmatrix} -\sqrt{2} \\ 3 \\ \sqrt{2} \\ 3 \end{bmatrix}$

○ $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\frac{1}{22}\begin{bmatrix} -\sqrt{2} \\ 3 \\ \sqrt{2} \\ -3 \end{bmatrix}$

○ $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\frac{1}{\sqrt{22}}\begin{bmatrix} -\sqrt{2} \\ 3 \\ \sqrt{2} \\ 3 \end{bmatrix}$

○ $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$, $\frac{1}{4}\begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

○ non of them

3. Which of the following systems of vectors is orthonormal: (choose the correct answer)

● non of them

○ $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\frac{1}{22}\begin{bmatrix} -\sqrt{2} \\ 3 \\ \sqrt{2} \\ -3 \end{bmatrix}$

○ $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\frac{1}{\sqrt{22}}\begin{bmatrix} -\sqrt{2} \\ 3 \\ \sqrt{2} \\ 3 \end{bmatrix}$

○ $\begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\frac{1}{\sqrt{22}}\begin{bmatrix} -\sqrt{2} \\ -3 \\ \sqrt{2} \\ 3 \end{bmatrix}$

4. Which of the following systems of vectors is orthonormal: (choose the correct answer)

- $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$
- ○ $\frac{1}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$
- ○ $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$
- ○ $\begin{bmatrix}\sqrt{2}\\0\\\sqrt{2}\\0\end{bmatrix}, \quad \begin{bmatrix}0\\\sqrt{2}\\0\\\sqrt{2}\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$
- ○ non of them

5. Which of the following systems of vectors is orthonormal: (choose the correct answer)

- $\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\2\\0\end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix}0\\2\\0\\1\end{bmatrix}, \quad \frac{1}{\sqrt{10}}\begin{bmatrix}2\\1\\-1\\-2\end{bmatrix}$
- ○ $\frac{1}{3}\begin{bmatrix}1\\0\\2\\0\end{bmatrix}, \quad \frac{1}{3}\begin{bmatrix}0\\2\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$
- ○ $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\0\\1\\0\end{bmatrix}, \quad \frac{1}{3}\begin{bmatrix}0\\2\\0\\1\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

○ $\frac{1}{4}\begin{bmatrix}\sqrt{2}\\0\\\sqrt{2}\\0\end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix}0\\\sqrt{2}\\0\\\sqrt{2}\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

○ non of them

6. Which of the following systems of vectors is orthonormal: (choose the correct answer)

● $\frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\2\\0\end{bmatrix}, \quad \frac{1}{2\sqrt{2}}\begin{bmatrix}0\\2\\0\\2\end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix}-2\\-2\\2\\2\end{bmatrix}$

○ $\frac{1}{4}\begin{bmatrix}2\\0\\2\\0\end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix}0\\2\\0\\2\end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$

○ $\frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\2\\0\end{bmatrix}, \quad \frac{-1}{2\sqrt{2}}\begin{bmatrix}0\\2\\0\\2\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

○ $\frac{1}{4}\begin{bmatrix}\sqrt{2}\\0\\\sqrt{2}\\0\end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix}0\\\sqrt{2}\\0\\\sqrt{2}\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

○ non of them

7. Polynomials

1. Find all the values of k for which the following system of polynomials

$$p_1(t) = t^3, \quad p_2(t) = 2t^3 + 4t^2 + 4,$$
$$p_3(t) = 2t^2 + 9t + k, \quad p_4(t) = t^3 + 8t^2 + k^2 + 7$$

is linearly independent: (choose the correct answer)

- • $k \neq 1, \quad k \neq -1$
- ○ $k \neq 2,$
- ○ $k = 1, \quad k = -1$
- ○ $k = 0$
- ○ non of them

2. Find all the values of k for which the following system of polynomials

$$p_1(t) = t^3, \quad p_2(t) = 2t^3 + 4t^2 + 4,$$
$$p_3(t) = 2t^2 + 9t + k, \quad p_4(t) = t^3 + 8t^2 + k^2 + 7$$

is linearly dependent: (choose the correct answer)

- • $k = 1, \quad k = -1$
- ○ $k = 2,$
- ○ $k \neq 1, \quad k \neq -1$
- ○ $k \neq 0$
- ○ non of them

Solution Determinant of the coefficient matrix is equal to $36(k^2 - 1)$

3. Find all the values of k for which the following system of polynomials

$$p_1(t) = 2t^3 + 3t^2, \quad p_2(t) = 5t^2 + (k-3)t,$$
$$p_3(t) = 5t^2 + (k-3)t + 9, \quad p_4(t) = 4t^3 + 6t^2 + (k^2 - 4)t$$

is linearly independent: (choose the correct answer)

- $k \neq 2, \quad k \neq -2$
- ○ $k \neq 2,$
- ○ $k = 2, \quad k = -2$
- ○ $k = 3$
- ○ non of them

4. Find all the values of k for which the following system of polynomials

$$p_1(t) = 2t^3 + 3t^2, \quad p_2(t) = 5t^2 + (k-3)t,$$

$$p_3(t) = 5t^2 + (k-3)t + 9, \quad p_4(t) = 4t^3 + 6t^2 + (k^2-4)t$$

is linearly dependent: (choose the correct answer)

- $k = 2, \quad k = -2$
- ○ $k \neq 0,$
- ○ $k \neq 2, \quad k \neq -2$
- ○ $k = 3$
- ○ non of them

SOLUTION Determinant of the coefficient matrix is equal to $90(4-k^2)$.

5. Find all the values of k for which the following system of polynomials

$$p_1(t) = (k^2-9)t^3, \quad p_2(t) = 4t^2 + 8t + k - 1,$$

$$p_3(t) = 8t^2 + 15t + 15, \quad p_4(t) = 2t^2 + 4t + k$$

is linearly independent: (choose the correct answer)

- non of them
- ○ $k \neq 3,$
- ○ $k = 3, \quad k = -3$
- ○ $k = 0$
- ○ $k \neq 3, \quad k \neq -3$

6. Find all the values of k for which the following system of polynomials

$$p_1(t) = (k^2 - 9)t^3, \quad p_2(t) = 4t^2 + 8t + k - 1,$$
$$p_3(t) = 8t^2 + 15t + 15, \quad p_4(t) = 2t^2 + 4t + k$$

is linearly dependent: (choose the correct answer)

- ● non of them
- ○ $k = 3,$
- ○ $k \neq 3, \quad k \neq -3$
- ○ $k = 0$
- ○ $k = 3, \quad k = -3$

SOLUTION Determinant of the coefficient matrix is equal to $-2(k^2 - 9)(k + 1)$.

8. Change of basis

1. Let
$$\mathcal{A} = \{\overrightarrow{a}_1, \overrightarrow{a}_2\}, \qquad \mathcal{B} = \{\overrightarrow{b}_1, \overrightarrow{b}_2\}$$
be two bases of a vector space V. Suppose that
$$\overrightarrow{a}_1 = 2\overrightarrow{b}_1 + 3\overrightarrow{b}_2, \qquad \overrightarrow{a}_1 = 5\overrightarrow{b}_1 + 7\overrightarrow{b}_2.$$
Let also
$$\overrightarrow{x} = \overrightarrow{b}_1 - \overrightarrow{b}_2.$$
Then the change-of-coordinates matrix $\mathcal{P}_{\mathcal{A}\to\mathcal{B}}$ from the basis $\mathcal{A}$ to the basis $\mathcal{B}$ and $[\overrightarrow{x}]_{\mathcal{A}}$ are given by: (choose the correct answer)

- ●
$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} 12 \\ -5 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

- ○ non of them

2. Let
$$\mathcal{A} = \{\overrightarrow{a}_1, \overrightarrow{a}_2\}, \qquad \mathcal{B} = \{\overrightarrow{b}_1, \overrightarrow{b}_2\}$$
be two bases of a vector space V. Suppose that
$$\overrightarrow{a}_1 = 2\overrightarrow{b}_1 + 3\overrightarrow{b}_2, \qquad \overrightarrow{a}_1 = 5\overrightarrow{b}_1 + 7\overrightarrow{b}_2.$$
Let also
$$\overrightarrow{x} = \overrightarrow{b}_1 - \overrightarrow{b}_2.$$
Then the change-of-coordinates matrix $\mathcal{P}_{\mathcal{A}\to\mathcal{B}}$ from the basis $\mathcal{A}$ to the basis $\mathcal{B}$ and $[\overrightarrow{x}]_{\mathcal{A}}$ are given by: (choose the correct answer)

- ● non of them

- ○
$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} 12 \\ -5 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -10 \\ 7 \end{bmatrix}$$

3. Let
$$\mathcal{A} = \{\overrightarrow{a}_1, \overrightarrow{a}_2\}, \qquad \mathcal{B} = \{\overrightarrow{b}_1, \overrightarrow{b}_2\}$$
be two bases of a vector space V. Suppose that
$$\overrightarrow{a}_1 = 2\overrightarrow{b}_1 + 3\overrightarrow{b}_2, \qquad \overrightarrow{a}_1 = 5\overrightarrow{b}_1 + 7\overrightarrow{b}_2.$$
Let also
$$\overrightarrow{x} = \overrightarrow{b}_1 - \overrightarrow{b}_2.$$
Then the change-of-coordinates matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{A}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{A}$ and $[\overrightarrow{x}]_{\mathcal{A}}$ are given by: (choose the correct answer)

- ● non of them

- ○
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} 12 \\ -5 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -10 \\ 7 \end{bmatrix}$$

4. Let

$$\mathcal{A} = \{\overrightarrow{a}_1, \overrightarrow{a}_2\}, \qquad \mathcal{B} = \{\overrightarrow{b}_1, \overrightarrow{b}_2\}$$

be two bases of a vector space V. Suppose that

$$\overrightarrow{a}_1 = 2\overrightarrow{b}_1 + 3\overrightarrow{b}_2, \qquad \overrightarrow{a}_1 = 5\overrightarrow{b}_1 + 7\overrightarrow{b}_2.$$

Let also

$$\overrightarrow{x} = \overrightarrow{b}_1 - \overrightarrow{b}_2.$$

Then the change-of-coordinates matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{A}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{A}$ and $[\overrightarrow{x}]_{\mathcal{A}}$ are given by: (choose the correct answer)

- ●
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = -\begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} 12 \\ -5 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

- ○
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

- ○ non of them

5. Let

$$\mathcal{A} = \{\overrightarrow{a}_1, \overrightarrow{a}_2\}, \qquad \mathcal{B} = \{\overrightarrow{b}_1, \overrightarrow{b}_2\}$$

be two bases of a vector space V. Suppose that

$$\overrightarrow{a}_1 = 2\overrightarrow{b}_1 + 3\overrightarrow{b}_2, \qquad \overrightarrow{a}_1 = 5\overrightarrow{b}_1 + 7\overrightarrow{b}_2.$$

Let also

$$\overrightarrow{x} = \overrightarrow{a}_1 - \overrightarrow{a}_2.$$

Then the change-of-coordinates matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{A}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{A}$ and $[\overrightarrow{x}]_{\mathcal{B}}$ are given by: (choose the correct answer)

- ●
$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = -\begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

○ non of them

6. Let

$$\mathcal{A} = \{\overrightarrow{a}_1, \overrightarrow{a}_2\}, \qquad \mathcal{B} = \{\overrightarrow{b}_1, \overrightarrow{b}_2\}$$

be two bases of a vector space V. Suppose that

$$\overrightarrow{a}_1 = 2\overrightarrow{b}_1 + 3\overrightarrow{b}_2, \qquad \overrightarrow{a}_1 = 5\overrightarrow{b}_1 + 7\overrightarrow{b}_2.$$

Let also

$$\overrightarrow{x} = \overrightarrow{a}_1 - \overrightarrow{a}_2.$$

Then the change-of-coordinates matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{A}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{A}$ and $[\overrightarrow{x}]_{\mathcal{B}}$ are given by: (choose the correct answer)

● non of them

○

$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{B}\to\mathcal{A}} = -\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

9. Geometric transformations

1. The eigenvalues and eigenvectors of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is called reflection through the x_1-axis are given by: (choose the correct answer)

- ● $$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- ○ $$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
- ○ $$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$
- ○ $$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
- ○ non of them

Solution The matrix of the transformation is
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2. The eigenvalues and eigenvectors of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is called reflection through the x_2-axis are given by: (choose the correct answer)

- ● $$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- ○ $$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

○
$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

○
$$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

○ non of them

Solution The matrix of the transformation is
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. The eigenvalues and eigenvectors of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is called reflection through the line $x_2 = x_1$ are given by: (choose the correct answer)

●
$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

○
$$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

○
$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

○
$$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

○ non of them

Solution The matrix of the transformation is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

4. The eigenvalues and eigenvectors of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is called reflection through the line $x_2 = -x_1$ are given by: (choose the correct answer)

- $$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- ○ $$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- ○ $$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- ○ $$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

- ○ non of them

Solution The matrix of the transformation is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

5. The eigenvalues and eigenvectors of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is called reflection through the origin are given by: (choose the correct answer)

- $$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- $$\lambda_1 = 1, \quad \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- $$\lambda_1 = 1, \quad \overrightarrow{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- $$\lambda_1 = -1, \quad \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \overrightarrow{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

- non of them

SOLUTION The matrix of the transformation is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

6. The eigenvalues and eigenvectors of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is called reflection through the line $x_2 = -x_1$ are given by: (choose the correct answer)

- ● non of them

- $$\lambda_1 = -1, \quad \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- $$\lambda_1 = 1, \quad \overrightarrow{v}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \overrightarrow{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- $$\lambda_1 = -1, \quad \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \overrightarrow{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

- $$\lambda_1 = 1, \quad \overrightarrow{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \lambda_2 = 1, \quad \overrightarrow{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution The matrix of the transformation is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

10. Rotation

1. The image of the vector $\begin{bmatrix}1\\2\end{bmatrix}$ under the rotation on $\frac{\pi}{6}$ in the counterclockwise direction is given by: (choose the correct answer)

- ● $$\begin{bmatrix}\frac{\sqrt{3}}{2}-1\\ \frac{1}{2}+\sqrt{3}\end{bmatrix}$$
- ○ $$\begin{bmatrix}\frac{\sqrt{2}}{2}-\sqrt{2}\\ \frac{\sqrt{2}}{2}+\sqrt{2}\end{bmatrix}$$
- ○ $$\begin{bmatrix}\frac{1}{2}-\sqrt{3}\\ \frac{\sqrt{3}}{2}+1\end{bmatrix}$$
- ○ $$\begin{bmatrix}\sqrt{3}-\frac{1}{2}\\ 1+\frac{\sqrt{3}}{2}\end{bmatrix}$$
- ○ $$\begin{bmatrix}\sqrt{2}-\frac{\sqrt{2}}{2}\\ \sqrt{2}+\frac{\sqrt{2}}{2}\end{bmatrix}$$

Solution The rotation matrix is given by

$$\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

2. The image of the vector $\begin{bmatrix}1\\2\end{bmatrix}$ under the rotation on $\frac{\pi}{4}$ in the counterclockwise direction is given by: (choose the correct answer)

- ● $$\begin{bmatrix}\frac{\sqrt{2}}{2}-\sqrt{2}\\ \frac{\sqrt{2}}{2}+\sqrt{2}\end{bmatrix}$$

- ○
$$\begin{bmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \end{bmatrix}$$

- ○
$$\begin{bmatrix} \frac{1}{2} - \sqrt{3} \\ \frac{\sqrt{3}}{2} + 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} \sqrt{3} - \frac{1}{2} \\ 1 + \frac{\sqrt{3}}{2} \end{bmatrix}$$

- ○
$$\begin{bmatrix} \sqrt{2} - \frac{\sqrt{2}}{2} \\ \sqrt{2} + \frac{\sqrt{2}}{2} \end{bmatrix}$$

Solution The rotation matrix is given by

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

3. The image of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ under the rotation on $\frac{\pi}{3}$ in the counterclockwise direction is given by: (choose the correct answer)

- ●
$$\begin{bmatrix} \frac{1}{2} - \sqrt{3} \\ \frac{\sqrt{3}}{2} + 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \end{bmatrix}$$

- ○
$$\begin{bmatrix} \frac{\sqrt{2}}{2} - \sqrt{2} \\ \frac{\sqrt{2}}{2} + \sqrt{2} \end{bmatrix}$$

- ○
$$\begin{bmatrix} \sqrt{3} - \frac{1}{2} \\ 1 + \frac{\sqrt{3}}{2} \end{bmatrix}$$

○

$$\begin{bmatrix} \sqrt{2} - \frac{\sqrt{2}}{2} \\ \sqrt{2} + \frac{\sqrt{2}}{2} \end{bmatrix}$$

Solution The rotation matrix is given by

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

4. The image of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ under the rotation on $\frac{\pi}{6}$ in the counterclockwise direction is given by: (choose the correct answer)

●

$$\begin{bmatrix} \sqrt{3} - \frac{1}{2} \\ 1 + \frac{\sqrt{3}}{2} \end{bmatrix}$$

○

$$\begin{bmatrix} \frac{\sqrt{2}}{2} - \sqrt{2} \\ \frac{\sqrt{2}}{2} + \sqrt{2} \end{bmatrix}$$

○

$$\begin{bmatrix} \frac{1}{2} - \sqrt{3} \\ \frac{\sqrt{3}}{2} + 1 \end{bmatrix}$$

○

$$\begin{bmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \end{bmatrix}$$

○

$$\begin{bmatrix} 1 - \frac{\sqrt{3}}{2} \\ \sqrt{3} + \frac{1}{2} \end{bmatrix}$$

Solution The rotation matrix is given by

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

5. The image of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ under the rotation on $\frac{\pi}{4}$ in the counterclockwise direction is given by: (choose the correct answer)

- ●
$$\begin{bmatrix}\sqrt{2}-\frac{\sqrt{2}}{2}\\ \sqrt{2}+\frac{\sqrt{2}}{2}\end{bmatrix}$$

- ○
$$\begin{bmatrix}\frac{\sqrt{3}}{2}-1\\ \frac{1}{2}+\sqrt{3}\end{bmatrix}$$

- ○
$$\begin{bmatrix}\frac{1}{2}-\sqrt{3}\\ \frac{\sqrt{3}}{2}+1\end{bmatrix}$$

- ○
$$\begin{bmatrix}1-\frac{\sqrt{3}}{2}\\ \sqrt{3}+\frac{1}{2}\end{bmatrix}$$

- ○
$$\begin{bmatrix}\frac{\sqrt{2}}{2}-\sqrt{2}\\ \frac{\sqrt{2}}{2}+\sqrt{2}\end{bmatrix}$$

Solution The rotation matrix is given by

$$\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

6. The image of the vector $\begin{bmatrix}2\\1\end{bmatrix}$ under the rotation on $\frac{\pi}{3}$ in the counterclockwise direction is given by: (choose the correct answer)

- ●
$$\begin{bmatrix}1-\frac{\sqrt{3}}{2}\\ \sqrt{3}+\frac{1}{2}\end{bmatrix}$$

- ○
$$\begin{bmatrix}\frac{\sqrt{3}}{2}-1\\ \frac{1}{2}+\sqrt{3}\end{bmatrix}$$

- ○
$$\begin{bmatrix}\frac{\sqrt{2}}{2}-\sqrt{2}\\ \frac{\sqrt{2}}{2}+\sqrt{2}\end{bmatrix}$$

- $$\begin{bmatrix} \sqrt{3} - \frac{1}{2} \\ 1 + \frac{\sqrt{3}}{2} \end{bmatrix}$$

- $$\begin{bmatrix} \sqrt{2} - \frac{\sqrt{2}}{2} \\ \sqrt{2} + \frac{\sqrt{2}}{2} \end{bmatrix}$$

Solution The rotation matrix is given by

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

11. Coordinates in the basis

1. The coordinates of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the basis

$$\overrightarrow{e}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \overrightarrow{e}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

2. The coordinates of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the basis

$$\overrightarrow{e}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \overrightarrow{e}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

3. The coordinates of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the basis

$$\overrightarrow{e}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \overrightarrow{e}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$
- ○
$$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$
- ○
$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$
- ○
$$\begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$
- ○
$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

4. The coordinates of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the basis

$$\overrightarrow{e}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overrightarrow{e}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

are given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

5. The coordinates of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the basis

$$\overrightarrow{e}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overrightarrow{e}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \overrightarrow{e}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

are given by: (choose the correct answer)

- ● $$\begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

- ○ $$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

- ○ $$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

- ○ $$\begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$

- ○ $$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

6. The coordinates of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the basis

$$\overrightarrow{e}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \overrightarrow{e}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overrightarrow{e}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overrightarrow{e}_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

are given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$
- ○
$$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$
- ○
$$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$
- ○
$$\begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$
- ○
$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

12. Diagonalization

1. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & 3 & \frac{1}{2} \\ 0 & 0 & 2 \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

be the matrix that consists of eigenvectors of the matrix A. Then the diagonalisation of A is given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 1 \end{bmatrix}$,

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 2 \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

be the matrix that consists of eigenvectors of the matrix A. Then the diagonalisation of A is given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

be the matrix that consists of eigenvectors of the matrix A. Then the diagonalisation of A is given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ \frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 3 \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

be the matrix that consists of eigenvectors of the matrix A. Then the diagonalisation of A is given by: (choose the correct answer)

- ● $$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
- ○ $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
- ○ $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- ○ $$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- ○ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

be the matrix that consists of eigenvectors of the matrix A. Then the diagonalisation of A is given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ○
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

6. Consider the matrix
$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Let
$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$
be the matrix that consists of eigenvectors of the matrix A. Then the diagonalisation of A is given by: (choose the correct answer)

- ●
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

13. Quadratic forms

1. The quadratic form

$$Q = -2x^2 + 2xz - 4y^2 - 2z^2$$

is: (choose the correct answer)

- ● negative definite
- ○ positive definite
- ○ negative semidefinite
- ○ positive semidefinite
- ○ indefinite

Eigenvalues $-4, -3, -1$

2. The quadratic form

$$Q = 2x^2 - 2xz + 4y^2 + 2z^2$$

is: (choose the correct answer)

- ● positive definite
- ○ negative definite
- ○ negative semidefinite
- ○ positive semidefinite
- ○ indefinite

Eigenvalues $4, 3, 1$

3. The quadratic form

$$Q = -x^2 + 4xz - 5y^2 - 4z^2$$

is: (choose the correct answer)

- ● negative semidefinite
- ○ positive definite
- ○ negative definite

- ○ positive semidefinite
- ○ indefinite

Eigenvalues $-5, -5, 0$

4. The quadratic form

$$Q = x^2 - 4xz + 3y^2 + z^2$$

is: (choose the correct answer)

- ● indefinite
- ○ positive definite
- ○ negative semidefinite
- ○ positive semidefinite
- ○ negative definite

Eigenvalues $3, 3, -1$

5. The quadratic form

$$Q = x^2 - 4xz + 6y^2 + z^2$$

is: (choose the correct answer)

- ● indefinite
- ○ positive definite
- ○ negative semidefinite
- ○ positive semidefinite
- ○ negative definite

Eigenvalues $6, 3, -1$

6. The quadratic form

$$Q = x^2 - 4xz + 6y^2 + 4z^2$$

is: (choose the correct answer)

- ● positive semidefinite

- ○ positive definite
- ○ negative semidefinite
- ○ negative definite
- ○ indefinite

Eigenvalues $6, 5, 0$

14. Least square problem

1. A certain experiment produces the data

$$\{(0,2),(-1,-1),(-2,0),(1,1)\}.$$

The least square curve $y=\beta_1 x+\beta_2 x^2$ is given by: (choose the correct answer)

- • $y=\frac{9}{11}x+\frac{4}{11}x^2$
- ○ $y=-\frac{9}{11}x+\frac{4}{11}x^2$
- ○ $y=36x-16x^2$
- ○ $y=-36x+16x^2$
- ○ non of them

Solution

$$X^TX=\begin{bmatrix}6 & -8\\ -8 & 18\end{bmatrix},\quad (X^TX)^{-1}X^Ty=\begin{bmatrix}\frac{9}{11}\\ \frac{4}{11}\end{bmatrix}$$

2. A certain experiment produces the data

$$\{(0,2),(1,-1),(-2,0),(-1,1)\}.$$

The least square curve $y=\beta_1 x+\beta_2 x^2$ is given by: (choose the correct answer)

- • $y=-\frac{9}{11}x-\frac{4}{11}x^2$
- ○ $y=\frac{9}{11}x-\frac{4}{11}x^2$
- ○ $y=36x-16x^2$
- ○ $y=-36x+16x^2$
- ○ non of them

Solution

$$X^TX=\begin{bmatrix}6 & -8\\ -8 & 18\end{bmatrix},\quad (X^TX)^{-1}X^Ty=\begin{bmatrix}\frac{-9}{11}\\ \frac{-4}{11}\end{bmatrix}$$

3. A certain experiment produces the data

$$\{(0,2),(1,-1),(2,0),(-1,1)\}.$$

The least square curve $y=\beta_1 x+\beta_2 x^2$ is given by: (choose the correct answer)

- • $y=-\frac{9}{11}x+\frac{4}{11}x^2$
- ○ $y=\frac{9}{11}x-\frac{4}{11}x^2$

- ○ $y = -36x + 16x^2$
- ○ $y = 36x + 16x^2$
- ○ non of them

Solution

$$X^T X = \begin{bmatrix} 6 & 8 \\ 8 & 18 \end{bmatrix}, \quad (X^T X)^{-1} X^T y = \begin{bmatrix} \frac{-9}{11} \\ \frac{4}{11} \end{bmatrix}$$

4. A certain experiment produces the data

$$\{(0,2),(-1,-1),(-2,0),(1,1)\}.$$

The least straight line $y = \beta_0 + \beta_1 x$ is given by: (choose the correct answer)

- ● $y = \frac{4}{5} + \frac{3}{5}x$
- ○ $y = -\frac{4}{5} - \frac{3}{5}x$
- ○ $y = 16 + 12x$
- ○ $y = -16 - 12x$
- ○ non of them

Solution

$$X^T X = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}, \quad (X^T X)^{-1} X^T y = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

5. A certain experiment produces the data

$$\{(0,2),(-1,-1),(2,0),(1,1)\}.$$

The least straight line $y = \beta_0 + \beta_1 x$ is given by: (choose the correct answer)

- ● $y = \frac{2}{5} + \frac{1}{5}x$
- ○ $y = -\frac{2}{5} + \frac{1}{5}x$
- ○ $y = 8 + 4x$
- ○ $y = -8 - 4x$
- ○ non of them

Solution

$$X^T X = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}, \quad , \quad (X^T X)^{-1} X^T y = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$

6. A certain experiment produces the data

$$\{(0, 2), (1, -1), (2, 0), (1, 1)\}.$$

The least straight line $y = \beta_0 + \beta_1 x$ is given by: (choose the correct answer)

- ● $y = \frac{3}{2} - x$
- ○ $y = 12 - 8x$
- ○ $y = \frac{3}{2} + x$
- ○ $y = -12 + 8x$
- ○ non of them

Solution

$$X^T X = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix}, \quad (X^T X)^{-1} X^T y = \begin{bmatrix} \frac{3}{2} \\ -1 \end{bmatrix}$$

15. Orthogonal projection

1. Let

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

The orthogonal projection of $\vec{y}$ on $H = \text{spn}\{\vec{u}_1, \vec{u}_2\}$ is given by: (choose the correct answer)

- ●
$$\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

- ○
$$2\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

- ○
$$-\frac{1}{2}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

- ○ non of them

Solution In this problem and all others the vectors are orthogonal and has length $\sqrt{2}$. If we denote by U the matrix that has columns $\vec{u}_1$ and $\vec{u}_2$, then the matrix $\frac{1}{2}UU^T$ is the projection matrix, see the pensum book, page 353, Theorem 10. Therefore

$$\frac{1}{2}UU^T\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

2. Let

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

The orthogonal projection of $\overrightarrow{y}$ on $H = \text{spn}\{\overrightarrow{u}_1, \overrightarrow{u}_2\}$ is given by: (choose the correct answer)

- non of them

○

$$2\begin{bmatrix}1\\4\\0\end{bmatrix}$$

○

$$\frac{1}{2}\begin{bmatrix}-1\\4\\0\end{bmatrix}$$

○

$$-\frac{1}{2}\begin{bmatrix}-1\\4\\0\end{bmatrix}$$

○

$$\frac{1}{2}\begin{bmatrix}1\\4\\0\end{bmatrix}$$

Solution

$$\frac{1}{2}UU^T\overrightarrow{y} = \begin{bmatrix}1\\4\\0\end{bmatrix}$$

3. Let

$$\overrightarrow{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \overrightarrow{u}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}, \quad \overrightarrow{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix}.$$

The orthogonal projection of $\overrightarrow{y}$ on $H = \text{spn}\{\overrightarrow{u}_1, \overrightarrow{u}_2\}$ is given by: (choose the correct answer)

- non of them

○

$$2\begin{bmatrix}-1\\0\\3\end{bmatrix}$$

- ○

$$\begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$$

- ○

$$-\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

- ○

$$\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

Solution

$$\frac{1}{2}UU^T\overrightarrow{y} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

4. Let

$$\overrightarrow{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \overrightarrow{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \overrightarrow{y} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

The orthogonal projection of $\overrightarrow{y}$ on $H = \text{spn}\{\overrightarrow{u}_1, \overrightarrow{u}_2\}$ is given by: (choose the correct answer)

- ● non of them

- ○

$$2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

- ○

$$\begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$$

○

$$-\frac{1}{2}\begin{bmatrix}-1\\0\\3\end{bmatrix}$$

○

$$\frac{1}{2}\begin{bmatrix}1\\0\\3\end{bmatrix}$$

Solution

$$\frac{1}{2}UU^T\overrightarrow{y}=\begin{bmatrix}1\\0\\3\end{bmatrix}$$

5. Let

$$\overrightarrow{u}_1=\begin{bmatrix}-1\\0\\-1\end{bmatrix},\quad \overrightarrow{u}_2=\begin{bmatrix}1\\0\\-1\end{bmatrix},\quad \overrightarrow{y}=\begin{bmatrix}-1\\4\\-3\end{bmatrix}.$$

The orthogonal projection of $\overrightarrow{y}$ on $H=\text{spn}\{\overrightarrow{u}_1,\overrightarrow{u}_2\}$ is given by: (choose the correct answer)

●

$$\begin{bmatrix}-1\\0\\-3\end{bmatrix}$$

○

$$2\begin{bmatrix}1\\0\\3\end{bmatrix}$$

○

$$\frac{1}{2}\begin{bmatrix}-1\\0\\-3\end{bmatrix}$$

○

$$-\frac{1}{2}\begin{bmatrix}-1\\0\\3\end{bmatrix}$$

○ non of them

Solution

$$\frac{1}{2}UU^T\overrightarrow{y} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$$

6. Let

$$\overrightarrow{u}_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad \overrightarrow{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \overrightarrow{y} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}.$$

The orthogonal projection of $\overrightarrow{y}$ on $H = \text{spn}\{\overrightarrow{u}_1, \overrightarrow{u}_2\}$ is given by: (choose the correct answer)

- ● non of them
- ○ $$2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$
- ○ $$\begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$$
- ○ $$\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$
- ○ $$\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

Solution

$$\frac{1}{2}UU^T\overrightarrow{y} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

16. Rank theorem

1. Let the parameter solution of an equation $A\overrightarrow{x} = \overrightarrow{b}$, $\overrightarrow{b} \in \mathbb{R}^3$, is given by

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \\ 9 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} x_5.$$

Then the following is true: (choose the correct answer)

- ●

$$A \text{ is } (3 \times 5) - \text{matrix}, \quad \text{rank(A)} = 3, \quad \dim\big(\text{Row}(A)\big) = 3$$

- ○

$$A \text{ is } (5 \times 3) - \text{matrix}, \quad \text{rank(A)} = 3, \quad \dim\big(\text{Row}(A)\big) = 3$$

- ○

$$A \text{ is } (3 \times 5) - \text{matrix}, \quad \text{rank(A)} = 2, \quad \dim\big(\text{Row}(A)\big) = 3$$

- ○

$$A \text{ is } (5 \times 3) - \text{matrix}, \quad \text{rank(A)} = 2, \quad \dim\big(\text{Row}(A)\big) = 2$$

- ○ non of them

Solution. $\overrightarrow{b} \in \mathbb{R}^3$, therefore the matrix A has 3 rows. The solution vector $\overrightarrow{x} \in \mathbb{R}^5$, therefore the matrix A has 5 columns. There are 2 parameters, therefore $\dim(\text{Null}(A)) = 2$. By rank theorem we also have

$$\text{rank(A)} = \dim\big(\text{Row}(A)\big) = 5 - 2 = 3.$$

2. Let the parameter solution of an equation $A\overrightarrow{x} = \overrightarrow{b}$, $\overrightarrow{b} \in \mathbb{R}^4$, is given by

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ 5 \\ 8 \\ 4 \\ 5 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ 0 \\ -1 \\ 9 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 4 \end{bmatrix} x_5.$$

Then the following is true: (choose the correct answer)

- ● non of them
- ○ $$A \text{ is } (5 \times 4) - \text{matrix}, \quad \text{rank(A)} = 3, \quad \dim\big(\text{Row}(A)\big) = 3$$
- ○ $$A \text{ is } (3 \times 5) - \text{matrix}, \quad \text{rank(A)} = 3, \quad \dim\big(\text{Row}(A)\big) = 3$$
- ○ $$A \text{ is } (5 \times 3) - \text{matrix}, \quad \text{rank(A)} = 2, \quad \dim\big(\text{Row}(A)\big) = 3$$
- ○ $$A \text{ is } (4 \times 5) - \text{matrix}, \quad \text{rank(A)} = 3, \quad \dim\big(\text{Row}(A)\big) = 2$$

SOLUTION. $\overrightarrow{b} \in \mathbb{R}^4$, therefore the matrix A has 4 rows. The solution vector $\overrightarrow{x} \in \mathbb{R}^5$, therefore the matrix A has 5 columns. There are 3 parameters, therefore $\dim(\text{Null}(A)) = 3$. By rank theorem we also have

$$\text{rank(A)} = \dim\big(\text{Row}(A)\big) = 5 - 3 = 2.$$

3. Let the parameter solution of an equation $A\overrightarrow{x} = \overrightarrow{b}$, $\overrightarrow{b} \in \mathbb{R}^3$, is given by

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \\ 9 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} x_5.$$

Then the following is true: (choose the correct answer)

- ● $$A \text{ is } (3\times 5)-\text{matrix}, \quad \dim(\text{Null(A)})=2, \quad \dim\big(\text{Row}(A)\big)=3$$

- ○ $$A \text{ is } (5\times 3)-\text{matrix}, \quad \dim(\text{Null(A)})=2, \quad \dim\big(\text{Row}(A)\big)=3$$

- ○ $$A \text{ is } (3\times 5)-\text{matrix}, \quad \dim(\text{Null(A)})=3, \quad \dim\big(\text{Row}(A)\big)=3$$

- ○ $$A \text{ is } (5\times 3)-\text{matrix}, \quad \dim(\text{Null(A)})=3, \quad \dim\big(\text{Row}(A)\big)=2$$

- ○ non of them

SOLUTION. $\overrightarrow{b} \in \mathbb{R}^3$, therefore the matrix A has 3 rows. The solution vector $\overrightarrow{x} \in \mathbb{R}^5$, therefore the matrix A has 5 columns. There are 2 parameters, therefore $\dim(\text{Null}(A)) = 2$. By rank theorem we also have

$$\text{rank(A)} = \dim\big(\text{Row}(A)\big) = 5 - 2 = 3.$$

4. Let the parameter solution of an equation $A\overrightarrow{x} = \overrightarrow{b}$, $\overrightarrow{b} \in \mathbb{R}^4$, is given by

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ 5 \\ 8 \\ 4 \\ 5 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ 0 \\ -1 \\ 9 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 4 \end{bmatrix} x_5.$$

Then the following is true: (choose the correct answer)

- $$A \text{ is } (4\times 5)-\text{matrix}, \quad \dim(\text{Null(A)}) = 3, \quad \dim\big(\text{Row}(A)\big) = 2$$

- ○ $$A \text{ is } (5\times 4)-\text{matrix}, \quad \dim(\text{Null(A)}) = 2, \quad \dim\big(\text{Row}(A)\big) = 1$$

- ○ $$A \text{ is } (3\times 5)-\text{matrix}, \quad \dim(\text{Null(A)}) = 3, \quad \dim\big(\text{Col}(A)\big) = 2$$

- ○ $$A \text{ is } (5\times 3)-\text{matrix}, \quad \dim(\text{Null(A)}) = 2, \quad \dim\big(\text{Col}(A)\big) = 3$$

- ○ non of them

SOLUTION. $\overrightarrow{b} \in \mathbb{R}^4$, therefore the matrix A has 4 rows. The solution vector $\overrightarrow{x} \in \mathbb{R}^5$, therefore the matrix A has 5 columns. There are 3 parameters, therefore $\dim(\text{Null}(A)) = 3$. By rank theorem we also have

$$\text{rank(A)} = \dim\big(\text{Row}(A)\big) = 5 - 3 = 2.$$

5. Let the parameter solution of an equation $A\overrightarrow{x} = \overrightarrow{b}$, $\overrightarrow{b} \in \mathbb{R}^1$, is given by

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ -1 \\ 9 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} x_4.$$

Then the following is true: (choose the correct answer)

- $$A \text{ is } (1\times 5)-\text{matrix}, \quad \text{rank(A)} = 3, \quad \dim\big(\text{Null}(A)\big) = 2$$

○

$$A \text{ is } (5\times 3)-\text{matrix}, \quad \text{rank(A)}=3, \quad \dim\big(\text{Row}(A)\big)=3$$

○

$$A \text{ is } (2\times 5)-\text{matrix}, \quad \text{rank(A)}=3, \quad \dim\big(\text{Null}(A)\big)=2$$

○

$$A \text{ is } (5\times 1)-\text{matrix}, \quad \text{rank(A)}=3, \quad \dim\big(\text{Null}(A)\big)=2$$

○ non of them

Solution. $\overrightarrow{b} \in \mathbb{R}^1$, therefore the matrix A has 1 row. The solution vector $\overrightarrow{x} \in \mathbb{R}^5$, therefore the matrix A has 5 columns. There are 2 parameters, therefore $\dim(\text{Null}(A)) = 2$. By rank theorem we also have

$$\text{rank(A)} = \dim\big(\text{Row}(A)\big) = 5-2 = 1.$$

6. Let the parameter solution of an equation $A\overrightarrow{x} = \overrightarrow{b}$, $\overrightarrow{b} \in \mathbb{R}^1$, is given by

$$\overrightarrow{x} = \begin{bmatrix}1\\2\\2\\1\\1\end{bmatrix} x_1 + \begin{bmatrix}3\\-2\\-1\\9\\1\end{bmatrix} x_2 + \begin{bmatrix}-1\\4\\5\\6\\7\end{bmatrix} x_4.$$

Then the following is true: (choose the correct answer)

●

$$A \text{ is } (1\times 5)-\text{matrix}, \quad \text{rank(A)}=2, \quad \dim\big(\text{Null}(A)\big)=3$$

○

$$A \text{ is } (5\times 3)-\text{matrix}, \quad \text{rank(A)}=3, \quad \dim\big(\text{Row}(A)\big)=3$$

- ○ $$A \ \text{ is } \ (2\times 5)-\text{matrix}, \quad \text{rank(A)}=3, \quad \dim\big(\text{Null}(A)\big)=3$$

- ○ $$A \ \text{ is } (5\times 1)-\text{matrix}, \quad \text{rank(A)}=2, \quad \dim\big(\text{Null}(A)\big)=3$$

- ○ non of them

Solution. $\overrightarrow{b} \in \mathbb{R}^1$, therefore the matrix A has 1 row. The solution vector $\overrightarrow{x} \in \mathbb{R}^5$, therefore the matrix A has 5 columns. There are 3 parameters, therefore $\dim(\text{Null}(A))=3$. By rank theorem we also have

$$\text{rank(A)}=\dim\big(\text{Row}(A)\big)=5-3=2.$$

17. Basis of null space

1. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 1 & 4 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for Null(A) is given by: (choose the correct answer)

- ●

$$\overrightarrow{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

- ○

$$\overrightarrow{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

- ○

$$\overrightarrow{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

- ○

$$\overrightarrow{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

- ○ non of them

2. Let
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 2 & 4 & 0 & 0 & 0 \end{bmatrix}.$$
A basis for Null(A) is given by: (choose the correct answer)

- ● $$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ -3 \\ 1 \end{bmatrix}$$

- ○ $$\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -5 \\ 3 \\ 1 \end{bmatrix}$$

- ○ $$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -5 \end{bmatrix}$$

- ○ $$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$$

- ○ non of them

3. Let
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 5 & 0 & -5 \\ 2 & 4 & 0 & 0 & 0 \end{bmatrix}.$$
A basis for Null(A) is given by: (choose the correct answer)

- ●

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

- ○

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -5 \\ 3 \\ 1 \end{bmatrix}$$

- ○

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ -5 \end{bmatrix}$$

- ○

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$$

- ○ non of them

4. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 5 & 0 & -5 \\ 2 & 4 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for Null(A) is given by: (choose the correct answer)

- ●

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

○
$$\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

○
$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 6 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ -5 \end{bmatrix}$$

○
$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix}$$

○ non of them

5. Let
$$A = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

A basis for Null(A) is given by: (choose the correct answer)

●
$$\vec{v}_1 = \begin{bmatrix} -8 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

○
$$\vec{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

○

$$\overrightarrow{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

○

$$\overrightarrow{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

○ non of them

6. Let

$$A = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

A basis for Null(A) is given by: (choose the correct answer)

●

$$\overrightarrow{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -6 \\ 2 \end{bmatrix}$$

○

$$\overrightarrow{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

○

$$\overrightarrow{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

- ○ $$\overrightarrow{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

- ○ non of them

18. Kernel of a map

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal projection onto the line $x_1 = x_2$. Then the kernel of the map T is given by: (choose the correct answer)

- ● the straight line: $x_1 = -x_2$
- ○ the straight line: $x_1 = x_2$
- ○ the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- ○ the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- ○ non of them

Solution The matrix of the transformation is

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

The kernel is given by the span of the eigenvector, that is the straight line $x_1 = -x_2$.

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal projection onto the line $x_1 = -x_2$. Then the kernel of the map T is given by: (choose the correct answer)

- ● the straight line: $x_1 = x_2$
- ○ the straight line: $x_1 = -x_2$
- ○ the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- ○ the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- ○ non of them

SOLUTION The matrix of the transformation is

$$\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

The kernel is given by the span of the eigenvector, that is the straight line $x_1 = x_2$.

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal projection onto the line $x_1 = 0$. Then the kernel of the map T is given by: (choose the correct answer)

- ● non of them
- ○ the straight line: $x_1 = 0$
- ○ the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- ○ the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- ○ the straight line: $x_1 = x_2$

SOLUTION The matrix of the transformation is

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The kernel is given by the span of the eigenvector, that is the straight line $x_2 = 0$.

4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal projection onto the plane $x_3 = 0$. Then the kernel of the map T is given by: (choose the correct answer)

- ● the straight line: $x_1 = 0,\ x_2 = 0,\ x_3$ is arbitrary
- ○ the straight lines: $x_1 = t,\ x_2 = t,\ x_3$ and t are arbitrary
- ○ the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- ○ the vectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$
- ○ non of them

SOLUTION The matrix of the transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The kernel is given by the span of the eigenvector, that is the straight line $x_1 = 0,\ x_2 = 0,\ x_3$ is arbitrary.

5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal projection onto the plane $x_2 = 0$. Then the kernel of the map T is given by: (choose the correct answer)

- ● the straight line: $x_1 = 0,\ x_3 = 0,\ x_2$ is arbitrary
- ○ the straight lines: $x_1 = t,\ x_3 = t,\ x_2$ and t are arbitrary
- ○ the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- ○ the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$
- ○ non of them

SOLUTION The matrix of the transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The kernel is given by the span of the eigenvector, that is the straight line $x_1 = 0,\ x_3 = 0,\ x_2$ is arbitrary.

6. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal projection onto the plane $x_1 = 0$. Then the kernel of the map T is given by: (choose the correct answer)

- ● the straight line: $x_2 = 0,\ x_3 = 0,\ x_1$ is arbitrary
- ○ the straight lines: $x_2 = t,\ x_3 = t,\ x_1$ and t are arbitrary

- the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$
- non of them

SOLUTION The matrix of the transformation is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The kernel is given by the span of the eigenvector, that is the straight line $x_2 = 0,\ x_3 = 0,\ x_1$ is arbitrary.

19. Spectral decomposition

1. Let

$$A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}.$$

Then the spectral decomposition is given by: (choose the correct answer)

- ●
$$A = \frac{7}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- ○
$$A = 7\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- ○
$$A = 7\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- ○
$$A = \frac{7}{\sqrt{5}}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + \frac{2}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- ○ non of them

Solution

$$\lambda_1 = 7, \quad \overrightarrow{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 2, \quad \overrightarrow{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

2. Let

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}.$$

Then the spectral decomposition is given by: (choose the correct answer)

- ●
$$A = \frac{7}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

- ○
$$A = 7\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

○

$$A = 2\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + 7\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

○

$$A = \frac{2}{\sqrt{5}}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + \frac{7}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

○ non of them

Solution

$$\lambda_1 = 7, \quad \vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 2, \quad \vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

3. Let

$$A = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}.$$

Then the spectral decomposition is given by: (choose the correct answer)

●

$$A = -\frac{7}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

○

$$A = -7\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

○

$$A = 7\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

○

$$A = \frac{7}{\sqrt{5}}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + \frac{2}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

○ non of them

Solution

$$\lambda_1 = -7, \quad \vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \quad \vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

4. Let

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}.$$

Then the spectral decomposition is given by: (choose the correct answer)

- ●
$$A = -\frac{7}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} + \frac{3}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○
$$A = -7\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} + 3\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○
$$A = \frac{7}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} - \frac{3}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○
$$A = -\frac{7}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} + \frac{3}{\sqrt{10}}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○ non of them

Solution

$$\lambda_1 = -7, \quad \vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \qquad \lambda_2 = 3, \quad \vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

5. Let

$$A = \begin{bmatrix} -2 & 3 \\ 3 & 6 \end{bmatrix}.$$

Then the spectral decomposition is given by: (choose the correct answer)

- ●
$$A = \frac{7}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} - \frac{3}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

- ○
$$A = -7\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} + 3\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○
$$A = \frac{7}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} - \frac{3}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○
$$A = -\frac{7}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} + \frac{3}{\sqrt{10}}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○ non of them

Solution

$$\lambda_1 = 7, \quad \vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \lambda_2 = -3, \quad \vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix},$$

6. Let
$$A = \begin{bmatrix} 2 & -3 \\ -3 & -6 \end{bmatrix}.$$

Then the spectral decomposition is given by: (choose the correct answer)

- ●
$$A = -\frac{7}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \frac{3}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

- ○
$$A = -7\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} + 3\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○
$$A = \frac{7}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} - \frac{3}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○
$$A = -\frac{7}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} + \frac{3}{\sqrt{10}}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

- ○ non of them

Solution

$$\lambda_1 = -7, \quad \overrightarrow{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \qquad \lambda_2 = 3, \quad \overrightarrow{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

20. Diagonalizability

1. Suppose that a matrix A is diagonalizable and has the characteristic polynomial

$$\det(A - \lambda I) = \lambda^3(\lambda - 3)^2(\lambda + 2)(\lambda - 4)^2.$$

Let $(m \times n)$ be the size of the matrix A, d is the dimension of the eigen space corresponding to the eigenvalue $\lambda = 4$ and $p = \dim\big(\text{Null}\,(A)\big)$. Which of the following numbers correspond to the matrix A? (choose the correct answer)

- ●
$$m \times n = 8 \times 8, \quad d = 2, \quad p = 3$$
- ○
$$m \times n = 8 \times 4, \quad d = 3, \quad p = 2$$
- ○
$$m \times n = 4 \times 4, \quad d = 1, \quad p = 0$$
- ○
$$m \times n = 8 \times 8, \quad d = 3, \quad p = 0$$
- ○ non of them

2. Suppose that a matrix A is diagonalizable and has the characteristic polynomial

$$\det(A - \lambda I) = \lambda(\lambda - 3)^2(\lambda + 2)^2(\lambda - 4)^2.$$

Let $(m \times n)$ be the size of the matrix A, d is the dimension of the eigen space corresponding to the eigenvalue $\lambda = 4$ and $p = \dim\big(\text{Null}\,(A)\big)$. Which of the following numbers correspond to the matrix A? (choose the correct answer)

- ●
$$m \times n = 7 \times 7, \quad d = 2, \quad p = 1$$
- ○
$$m \times n = 7 \times 4, \quad d = 3, \quad p = 2$$

○

$$m \times n = 4 \times 4, \quad d = 1, \quad p = 0$$

○

$$m \times n = 7 \times 7, \quad d = 3, \quad p = 0$$

○ non of them

3. Suppose that a matrix A is diagonalizable and has the characteristic polynomial

$$\det(A - \lambda I) = \lambda^3(\lambda - 3)^2(\lambda + 2)(\lambda - 4)^2.$$

Let $(m \times n)$ be the size of the matrix A, d is the dimension of the eigen space corresponding to the eigenvalue $\lambda = 4$ and $p = (\text{rank}\,(A))$. Which of the following numbers correspond to the matrix A? (choose the correct answer)

●

$$m \times n = 8 \times 8, \quad d = 2, \quad p = 5$$

○

$$m \times n = 8 \times 4, \quad d = 3, \quad p = 3$$

○

$$m \times n = 4 \times 4, \quad d = 1, \quad p = 0$$

○

$$m \times n = 8 \times 8, \quad d = 3, \quad p = 5$$

○ non of them

4. Suppose that a matrix A is diagonalizable and has the characteristic polynomial

$$\det(A - \lambda I) = \lambda(\lambda - 3)^2(\lambda + 2)^2(\lambda - 4)^2.$$

Let $(m \times n)$ be the size of the matrix A, d is the dimension of the eigen space corresponding to the eigenvalue $\lambda = 4$ and $p = (\text{rank}\,(A))$. Which of the following numbers correspond to the matrix A? (choose the correct answer)

- ●

$$m \times n = 7 \times 7, \quad d = 2, \quad p = 6$$

- ○

$$m \times n = 7 \times 4, \quad d = 3, \quad p = 3$$

- ○

$$m \times n = 4 \times 4, \quad d = 1, \quad p = 0$$

- ○

$$m \times n = 7 \times 7, \quad d = 3, \quad p = 6$$

- ○ non of them

5. Suppose that a matrix A is diagonalizable and has the characteristic polynomial

$$\det(A - \lambda I) = \lambda^3(\lambda - 3)^2(\lambda + 2)(\lambda - 4)^2.$$

Let $(m \times n)$ be the size of the matrix A, d is the dimension of the eigen space corresponding to the eigenvalue $\lambda = -2$ and $p = \dim\big(\mathrm{Col}\,(A)\big)$. Which of the following numbers correspond to the matrix A? (choose the correct answer)

- ●

$$m \times n = 8 \times 8, \quad d = 1, \quad p = 5$$

- ○

$$m \times n = 8 \times 4, \quad d = 3, \quad p = 2$$

- ○

$$m \times n = 4 \times 4, \quad d = 1, \quad p = 0$$

- ○

$$m \times n = 8 \times 8, \quad d = 3, \quad p = 5$$

- ○ non of them

6. Suppose that a matrix A is diagonalizable and has the characteristic polynomial

$$\det(A - \lambda I) = \lambda(\lambda - 3)^2(\lambda + 2)^3(\lambda - 4)^2.$$

Let $(m \times n)$ be the size of the matrix A, d is the dimension of the eigen space corresponding to the eigenvalue $\lambda = -2$ and $p = \dim\big(\text{Col}\,(A)\big)$. Which of the following numbers correspond to the matrix A? (choose the correct answer)

- ●
$$m \times n = 8 \times 8, \quad d = 3, \quad p = 7$$

- ○
$$m \times n = 8 \times 4, \quad d = 3, \quad p = 2$$

- ○
$$m \times n = 4 \times 4, \quad d = 1, \quad p = 0$$

- ○
$$m \times n = 8 \times 8, \quad d = 2, \quad p = 7$$

- ○ non of them

21. Proof problem 1

1. Given $\overrightarrow{u} \in \mathbb{R}^n$ with $\overrightarrow{u}^T\overrightarrow{u} = 1$, let $P = \overrightarrow{u}\,\overrightarrow{u}^T$ and $Q = I - 2P$. Justify statements:

$$a) \quad P^2 = P;$$

$$b) \quad P^T = P;$$

$$c) \quad Q^2 = I.$$

Proof.

$$P^2 = PP = (\overrightarrow{u}\,\overrightarrow{u}^T)(\overrightarrow{u}\,\overrightarrow{u}^T) = \overrightarrow{u}(\overrightarrow{u}^T\overrightarrow{u})\overrightarrow{u}^T = \overrightarrow{u}\,\overrightarrow{u}^T.$$

$$P^T = (\overrightarrow{u}\,\overrightarrow{u}^T)^T = \overrightarrow{u}\,\overrightarrow{u}^T = P$$

$$Q^2 = QQ = (I - 2P)(I - 2P) = I - 2P - 2P + 4P^2 = I - 4P + 4P = I.$$

□

2. Let S be a maximal linearly independent subset of a vector space V. That is, S has the property that if a vector $\overrightarrow{v} \notin S$, then the new set $S \cup \overrightarrow{v}$, will no longer be linearly independent. Prove that S must be a basis for V.

Proof. To show that $S = \{\overrightarrow{b}_1, \ldots, \overrightarrow{b}_p\}$ is a basis, we need to show that

$$V = \operatorname{span}\{\overrightarrow{b}_1, \ldots, \overrightarrow{b}_p\}.$$

Take any vector $\overrightarrow{v} \in V$. Since $S \cup \overrightarrow{v}$ is linearly dependent, then

$$\overrightarrow{b}_k = \sum_{j \neq k} x_j \overrightarrow{b}_j + a\overrightarrow{v},$$

for some index k and $a \neq 0$. Because if it would be $a = 0$, then $S = \{\overrightarrow{b}_1, \ldots, \overrightarrow{b}_p\}$ would be linearly dependent. Therefore

$$\overrightarrow{v} = \frac{1}{a}\overrightarrow{b}_k - \sum_{j \neq k} \frac{x_j}{a}\overrightarrow{b}_j.$$

It shows that any $\overrightarrow{v} \in V$ is a linear combination of $S = \{\overrightarrow{b}_1, \ldots, \overrightarrow{b}_p\}$ and therfore

$$v \in \operatorname{span}\{\overrightarrow{b}_1, \ldots, \overrightarrow{b}_p\}.$$

□

3. Show that

a) if B is $(n \times p)$, then $\text{rank}(AB) \leq \text{rank}(A)$ [Hint: Explain why every vector in the column space of AB is in the column space of A.]

b) Show that if B is $(n \times p)$, then $\text{rank}(AB) \leq \text{rank}(B)$. [Hint: Use part (a) to study $\text{rank}(AB)^T$.]

Proof. Let us show a).

$$AB = [A\overrightarrow{b}_1 \dots A\overrightarrow{b}_p],$$

where each column in AB has the form

$$A\overrightarrow{b}_k = b_{1k}\overrightarrow{a}_1 + \ldots + b_{nk}\overrightarrow{a}_n \in \text{Col}(A)$$

and therefore is a linear combination of columns of A. Therefore any $\overrightarrow{x} \in \text{Col}(AB)$ is actually a vector from $\text{Col}(A)$. It shows that

$$\text{Col}(AB) \subset \text{Col}(A).$$

We conclude that

$$\text{rank}(AB) = \dim(\text{Col}(AB)) \leq \dim(\text{Col}(A)) = \text{rank}(A).$$

Let us show b). We know that $\text{rank}(A) = \text{rank}(A^T)$ since the number of linearly independent columns is equal to the number of linearly independent rows. Therefore by using *a*) we obtain

$$\text{rank}(AB) = \text{rank}(AB)^T = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B).$$

□

22. Proof problem 2

1. If

$$p(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n,$$

we define $p(A)$ to be the matrix formed by replacing each power of t in $p(t)$ by the corresponding power of A with ($A^0 = I$). That is,

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \ldots + c_n A^n.$$

Show that if λ is an eigenvalue of A, then one eigenvalue of $p(A)$ is $p(\lambda)$.

Proof. Let $A\overrightarrow{v} = \lambda\overrightarrow{v}$. Then

$$\begin{aligned} p(A)\overrightarrow{v} &= c_0 I\overrightarrow{v} + c_1 A\overrightarrow{v} + c_2 A^2\overrightarrow{v} + \ldots + c_n A^n\overrightarrow{v} \\ &= c_0\overrightarrow{v} + c_1\lambda\overrightarrow{v} + c_2\lambda^2\overrightarrow{v} + \ldots + c_n\lambda^n\overrightarrow{v} \\ &= (c_0 + c_1\lambda + c_2\lambda^2 + \ldots + c_n\lambda^n)\overrightarrow{v} \\ &= p(\lambda)\overrightarrow{v}. \end{aligned}$$

Thus $p(\lambda)$ is one eigenvalue of $p(A)$. □

2 Show that $I - A$ is invertible when all the eigenvalues of A are less than 1 in magnitude.

Proof. If we assume that $I - A$ is not invertible, that the matrix $I - A$ would has an eigenvalue equal to 0. Then

$$(I - A)\overrightarrow{v} = 0, \quad \Longleftrightarrow \quad A\overrightarrow{v} = \overrightarrow{v}$$

and A would have an eigenvalue equal to 1. □

3. Explain why an equation $A\overrightarrow{x} = \overrightarrow{b}$ has a solution if and only if $\overrightarrow{b}$ is orthogonal to all solutions of the equation $A^T\overrightarrow{x} = \overrightarrow{0}$.

Proof. Denote by $\overrightarrow{u}$ a solution to $A^T\overrightarrow{x} = \overrightarrow{0}$.

Assume that $A\overrightarrow{x} = \overrightarrow{b}$ has solution $\overrightarrow{v}$; that is $A\overrightarrow{x} = \overrightarrow{b}$. Then

$$\langle\overrightarrow{u}, \overrightarrow{b}\rangle = \langle\overrightarrow{u}, A\overrightarrow{v}\rangle = \langle A^T\overrightarrow{v}, \overrightarrow{x}\rangle = \langle\overrightarrow{0}, \overrightarrow{x}\rangle = 0.$$

We see that $\overrightarrow{b}$ is orthogonal to $\overrightarrow{u}$.

Assume that $\overrightarrow{b}$ is orthogonal to $\overrightarrow{u}$, where $\overrightarrow{u}$ is any solution to $A^T\overrightarrow{x} = \overrightarrow{0}$; that is $A^T\overrightarrow{u} = \overrightarrow{0}$. We see that $\overrightarrow{u} \in \text{Null}(A^T)$ and $\overrightarrow{b} \in \Big(\text{Null}(A^T)\Big)^{\perp}$. We know that (Theorem 3, page 337 from the book)

$$\Big(\text{Null}(A^T)\Big)^{\perp} = \text{Col}(A).$$

Thus $\overrightarrow{b} \in \text{Col}(A)$ and therefore there is $\overrightarrow{v}$ such that $A\overrightarrow{v} = \overrightarrow{b}$.

□