Problem 1 Solution

Consider the equation system:

$$\begin{aligned} x + z &= 1 \\ ax + y + z &= 1 \\ x + y + az &= -2 \end{aligned}$$

Part (a)

For what values of the parameter a will the system have exactly one solution?

Solution for Part (a)

The coefficient matrix A and its determinant $\det(A)$ are:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ a & 1 & 1 \\ 1 & 1 & a \end{pmatrix}$$

$$\det(A) = 1(1 \cdot a - 1 \cdot 1) - 0(a \cdot a - 1 \cdot 1) + 1(a \cdot 1 - 1 \cdot 1) = (a - 1) + (a - 1) = 2(a - 1)$$

The system has exactly one solution if $\det(A) \neq 0$, that is for all values of a except $a = 1$.

Part (b)

Are there any values of a where the system has no solutions?

Solution for Part (b)

We check the case where $\det(A) = 0$, which is when $a = 1$. We substitute $a = 1$ into the augmented matrix and perform Gaussian elimination.

$$[A|\mathbf{b}] \xrightarrow{a=1} \begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 1 & 1 & 1 & | & 1 \\ 1 & 1 & 1 & | & -2 \end{pmatrix}$$

Already here we see that there is no solution as the two last line in the matrix are identical but the right hand side is different. $x + y + z$ cannot both equal 1 and -2.

For those who do not notice this, the approach will be to do a Gaussian elimination.

1. $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$:

$$\begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & -3 \end{pmatrix}$$

2. $R_3 \leftarrow R_3 - R_2$:

$$\begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$$

The last row represents the equation $0x + 0y + 0z = -3$, which is a contradiction ($0 = -3$). Therefore, the system has no solutions when $a = 1$.

Part (c)

Show that for $a = 0$, the inverse of the coefficient matrix A is

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Solution for Part (c)

For $a = 0$, the coefficient matrix is $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. We show that A^{-1} is the inverse by verifying that the product AA^{-1} equals the identity matrix I.

$$\begin{aligned} AA^{-1} &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1(1)+0(-1)+1(1) & 1(-1)+0(1)+1(1) & 1(1)+0(1)+1(-1) \\ 0(1)+1(-1)+1(1) & 0(-1)+1(1)+1(1) & 0(1)+1(1)+1(-1) \\ 1(1)+1(-1)+0(1) & 1(-1)+1(1)+0(1) & 1(1)+1(1)+0(-1) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \end{aligned}$$

Since $AA^{-1} = I$, the given matrix is verified to be the inverse of A for $a = 0$.

Part (d)

Write the following equation system on matrix form. Use the result from part (c) to solve the system.

$$\begin{aligned} x + z &= 1 \\ y + z &= 1 \\ x + y &= -2 \end{aligned}$$

Solution for Part (d)

The system can be written in the matrix form $\mathbf{Ax} = \mathbf{b}$, where the coefficient matrix A corresponds to the case $a = 0$ from the original problem:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

The solution is found by multiplying the inverse matrix A^{-1} (from part c) by the constant vector $\mathbf{b}$:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1(1)-1(1)+1(-2) \\ -1(1)+1(1)+1(-2) \\ 1(1)+1(1)-1(-2) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \end{aligned}$$

The solution is $\mathbf{x = -1}$, $\mathbf{y = -1}$, and $\mathbf{z = 2}$.

Dynamic Programming Problem: Optimal Stopping

Part (a)

The Bellman equation for this problem is:

$$V_t(x_t) = \max_{a_t} \left\{ R(x_t, a_t) + \frac{1}{1+d} \cdot V_{t+1}(x_{t+1}) \right\}$$

There are two choices of a_t. If we choose $a_t = \textit{Stop}$, then $R(x_t, a_t) = x_t$ while $x_{t+1} = 0$ so $V_{t+1}(x_{t+1}) = 0$, thus when $a_t = \textit{Stop}$ the right hand side of the Bellman equation is x_t. Alternatively, if we choose $a_t = \textit{Stop}$, there is no immediate revard so the right hand side of the Bellman equation is $\frac{1}{1+d} \cdot V_{t+1}(x_{t+1})$. As we choose a_t to maximize the right hand side it follows that

$$V_t(x) = \max \left\{ x, \quad \frac{1}{1+d} V_{t+1}(x(1+r)) \right\}$$

Part (b): Case $r < d$

Show that if the value function at time $t+1$ is given by

$$V_{t+1}(x) = x$$

then $V_t(x)$ is also given by

$$V_t(x) = x$$

Determine the optimal policy (u_t^*) at time t based on the assumption $r < d$.

Solution Sketch for Part (b)

Substitute the hypothesis for V_{t+1} into the Bellman equation:

$$V_t(x) = \max \left\{ x, \quad \frac{1}{1+d} \left[x(1+r) \right] \right\}$$

The term x is the Value of Stopping. The second term is the Value of Continuing:

$$\text{Value of Continuing} = x \cdot \frac{1+r}{1+d}$$

Since $r < d$, we have $\frac{1+r}{1+d} < 1$. Therefore, $x \cdot \frac{1+r}{1+d} < x$.

$$V_t(x) = \max \left\{ x, \quad x \cdot \frac{1+r}{1+d} \right\} = x$$

This verifies the inductive step.

Since the Value of Stopping (x) is greater than the Value of Continuing $(x \cdot \frac{1+r}{1+d})$, the optimal policy is to $a_t^* = \text{Stop}$.

Problem 3:

Consider the equation system:

$$F_1: \quad p_1 x_1^{1/2} - 2p_2 x_2^{1/2} = 0$$
$$F_2: \quad p_1 x_1 + p_2 x_2 - m = 0$$

Assume that x_1, x_2, p_1, p_2, m are all non-negative. This system implicitly defines the demand functions $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.

Part (a)

Solution Sketch for Part (a)

Differentiation of $F_1 = p_1 x_1^{1/2} - 2p_2 x_2^{1/2} = 0$:

$$dF_1 = \left(\frac{\partial F_1}{\partial x_1}\right) dx_1 + \left(\frac{\partial F_1}{\partial x_2}\right) dx_2 + \left(\frac{\partial F_1}{\partial p_1}\right) dp_1 + \left(\frac{\partial F_1}{\partial p_2}\right) dp_2 + \left(\frac{\partial F_1}{\partial m}\right) dm = 0$$

$$\left(\frac{1}{2} p_1 x_1^{-1/2}\right) dx_1 + \left(-p_2 x_2^{-1/2}\right) dx_2 + (x_1^{1/2}) dp_1 + (-2x_2^{1/2}) dp_2 + (0) dm = 0$$

Total differentiation of $F_2 = p_1 x_1 + p_2 x_2 - m = 0$:

$$dF_2 = \left(\frac{\partial F_2}{\partial x_1}\right) dx_1 + \left(\frac{\partial F_2}{\partial x_2}\right) dx_2 + \left(\frac{\partial F_2}{\partial p_1}\right) dp_1 + \left(\frac{\partial F_2}{\partial p_2}\right) dp_2 + \left(\frac{\partial F_2}{\partial m}\right) dm = 0$$

$$(p_1) dx_1 + (p_2) dx_2 + (x_1) dp_1 + (x_2) dp_2 + (-1) dm = 0$$

Writing the system in matrix form $A \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} dR_1 \\ dR_2 \end{pmatrix}$:

$$\begin{pmatrix} \frac{p_1}{2\sqrt{x_1}} & -\frac{p_2}{\sqrt{x_2}} \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} -x_1^{1/2} dp_1 + 2x_2^{1/2} dp_2 \\ -x_1 dp_1 - x_2 dp_2 + dm \end{pmatrix}$$

Part (b)

A particular solution to the original system is $x_1 = 9$, $x_2 = 4$, $p_1 = 4$, $p_2 = 3$, and $m = 48$.

Use the system from Part (a) and **Cramer's Rule** to estimate the value of the cross-price derivative dx_1/dp_2 at this particular point. AS we only want to find dx_1/dp_2, we set $dp_1 = 0$, $dm = 0$)*

Solution Sketch for Part (b)

To find dx_1/dp_2, we set $dp_1 = 0$ and $dm = 0$ (holding p_1 and m constant) and divide the system by dp_2. The differential system becomes:

$$\begin{pmatrix} \frac{p_1}{2\sqrt{x_1}} & -\frac{p_2}{\sqrt{x_2}} \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{dx_1}{dp_2} \\ \frac{dx_2}{dp_2} \end{pmatrix} = \begin{pmatrix} 2x_2^{1/2} \\ -x_2 \end{pmatrix}$$

Substitute the values $p_1 = 4, p_2 = 3, x_1 = 9, x_2 = 4$ ($\sqrt{x_1} = 3$, $\sqrt{x_2} = 2$):

$$\begin{pmatrix} \frac{4}{2\cdot 3} & -\frac{3}{2} \\ 4 & 3 \end{pmatrix} \begin{pmatrix} dx_1/dp_2 \\ dx_2/dp_2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 \\ -4 \end{pmatrix} \implies \begin{pmatrix} 2/3 & -3/2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} dx_1/dp_2 \\ dx_2/dp_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

We can solve this using Cramers rule.

$$det(A) = \det \begin{pmatrix} 2/3 & -3/2 \\ 4 & 3 \end{pmatrix} = \left(\frac{2}{3}\right)(3) - \left(-\frac{3}{2}\right)(4) = 2 - (-6) = 8$$

Next we replace the first column of A with the constant vector $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$:

$$\det(A_1) = \det \begin{pmatrix} 4 & -3/2 \\ -4 & 3 \end{pmatrix} = (4)(3) - \left(-\frac{3}{2}\right)(-4) = 12 - 6 = 6$$

Finally, by Cramers rule

$$\frac{dx_1}{dp_2} = \frac{\det(A_1)}{J} = \frac{6}{8} = \frac{3}{4}$$

The estimated value of dx_1/dp_2 at the given point is $\mathbf{3/4}$.

Solution to Problem 4

(a) Solved:

$$I(t) = \int_0^\alpha xe^{tx}\,dx$$

$$\begin{aligned} u = x, &\quad v' = e^{tx} \\ \Rightarrow\ u' = 1, &\quad v = \frac{1}{t}e^{tx}. \end{aligned}$$

Using integration by parts,

$$\int_a^b uv'\,dx = [uv]_a^b - \int_a^b u'v\,dx,$$

we get

$$\begin{aligned} I(t) = \int_0^\alpha xe^{tx}\,dx &= \left[x\frac{1}{t}e^{tx}\right]_0^\alpha - \int_0^\alpha 1\cdot\frac{1}{t}e^{tx}\,dx \\ &= \frac{\alpha e^{t\alpha}}{t} - \frac{1}{t}\int_0^\alpha e^{tx}\,dx. \end{aligned}$$

(b) Solved:

$$J(t) = \int_0^\alpha e^{tx}\,dx$$

Then using the Leibniz rule on $J(t)$:

$$J'(t) = \int_0^\alpha \frac{\partial}{\partial t}e^{tx}dx = \int_0^\alpha xe^{tx}dx = I(t)$$

(c) Solved:

$$\begin{aligned} tI'(t) + I(t) &= t\hat{I}'(t) + \hat{I}(t) + t\frac{d}{dt}(\frac{C}{t}) + \frac{C}{t} \\ &= t\hat{I}'(t) + \hat{I}(t) - \frac{C}{t} + \frac{C}{t} \\ &= t\hat{I}'(t) + \hat{I}(t) = \alpha^2 e^{t\alpha} - \frac{\alpha e^{t\alpha}}{t} + \frac{e^{t\alpha}-1}{t^2} \end{aligned}$$

Where the last equality follows as $\hat{I}'(t)$ is a particular solution.

(d) Solved:

$$\begin{aligned} &I(1) = 1 \Rightarrow I(1) = \frac{(\alpha 1 - 1)e^\alpha + 1}{1^2} + \frac{C}{1} = 1 \\ &\Rightarrow (\alpha - 1)e^\alpha + 1 + C = 1 \\ &\Rightarrow C = 1 - 1 - (\alpha - 1)e^\alpha = e^\alpha(1-\alpha). \end{aligned}$$

(So, the particular solution is: $I(t) = \frac{(\alpha t - 1)e^{\alpha t}+1}{t^2} + \frac{e^\alpha(1-\alpha)}{t}$.)

Solution to Problem 5

A household consist of a husband H and a wife W. They share a joint utility function $U(c_H, c_W, l_H, l_W)$ over consumption c and leisure l of the husband and wife is represented by Equation (1):

$$U(c_H, c_W, l_H, l_W) = \frac{c_H^{1-\gamma}}{1-\gamma} + \frac{c_W^{1-\gamma}}{1-\gamma} + \frac{1}{1-\phi}\left[\alpha l_H + (1-\alpha) l_W\right]^{1-\phi} \tag{1}$$

That is, their utility in consumption is *separable* while their utility in leisure is *non-separable*, with a Pareto-weight α on the husband's leisure. They have a common budget constraint represented by Equation (2). The parameter κ represents the bargaining power of the husband. The budget constraint is:

$$w_H(1-l_H) + w_W(1-l_W) = \kappa p c_H + (1-\kappa) q c_W \tag{2}$$

with p and q being constant prices of the commodities bought by the husband and wife, respectively.

(a) Write up the Lagrangian for the utility maximization of this household and state all the associated *Lagrange conditions.*

Solution:

$$\mathcal{L} = \frac{c_H^{1-\gamma}}{1-\gamma} + \frac{c_W^{1-\gamma}}{1-\gamma} + \frac{1}{1-\phi}\left[\alpha l_H + (1-\alpha) l_W\right]^{1-\phi} - \lambda(w_H(1-l_H) + w_W(1-l_W) - \kappa p c_H - (1-\kappa) q c_W) \tag{3}$$

with associated Lagrange conditions:

$$\mathcal{L}_1 = c_H^{-\gamma} + \lambda\kappa p = 0 \tag{4}$$
$$\mathcal{L}_2 = c_W^{-\gamma} + \lambda(1-\kappa) q = 0 \tag{5}$$
$$\mathcal{L}_3 = \alpha(\alpha l_H + (1-\alpha) l_W)^{-\phi} + \lambda w_H = 0 \tag{6}$$
$$\mathcal{L}_4 = (1-\alpha)(\alpha l_H + (1-\alpha) l_W)^{-\phi} + \lambda w_W = 0 \tag{7}$$
$$w_H(1-l_H) + w_W(1-l_W) = \kappa p c_H + (1-\kappa) q c_W \tag{8}$$

(b) Assume an interior solution. Find an expression for the optimal ratio of consumption $\frac{c_H}{c_W}$ in terms of the parameters q, p and κ.

Solution: From Equations (4) and (5), we get:

$$\begin{aligned}
c_H^{-\gamma} &= -\lambda\kappa p \\
c_W^{-\gamma} &= -\lambda(1-\kappa) q \\
\implies \left(\frac{c_H}{c_W}\right)^{-\gamma} &= \frac{p}{q}\frac{\kappa}{(1-\kappa)} \\
\frac{c_H}{c_W} &= \left(\frac{q}{p}\frac{(1-\kappa)}{\kappa}\right)^{\frac{1}{\gamma}}
\end{aligned}$$

The interpretation (not asked for) is that if the price of the wife's commodities increases, the household buys more of the husband's commodities, and vice versa.

(c) Assume again an interior solution. Show that the first-order conditions imply:

$$\frac{\alpha}{1-\alpha} = \frac{w_H}{w_W} \tag{9}$$

Solution: From Equations (6) and (7), we get:

$$\begin{aligned}
\frac{\alpha}{1-\alpha}\frac{(\alpha l_H + (1-\alpha) l_W)^{-\phi}}{(\alpha l_H + (1-\alpha) l_W)^{-\phi}} &= \frac{w_H}{w_W} \\
\implies \frac{\alpha}{1-\alpha} &= \frac{w_H}{w_W}
\end{aligned}$$

Note (not asked for): if this does not hold, either one or both spouses have to be at a corner solution (with $l_j = 0$ and/or $l_j = 1$ for $j = H, W$).

(d) Assume that there is an interior solution, that is, that the condition from (b) holds. Show that this implies:

$$\alpha l_H + (1-\alpha) l_W = \left(-\lambda \frac{w_H}{\alpha}\right)^{-1/\phi} = \left(-\lambda \frac{w_W}{(1-\alpha)}\right)^{-1/\phi} \tag{10}$$

Solution: Use Equation (6):

$$\begin{aligned}
\alpha(\alpha l_H + (1-\alpha) l_W)^{-\phi} + \lambda w_H &= 0 \\
\alpha(\alpha l_H + (1-\alpha) l_W)^{-\phi} &= -\lambda w_H \\
(\alpha l_H + (1-\alpha) l_W)^{-\phi} &= -\lambda \frac{w_H}{\alpha} \\
\alpha l_H + (1-\alpha) l_W &= \left(-\lambda \frac{w_H}{\alpha}\right)^{-1/\phi}
\end{aligned}$$

Using Equation (7) would yield the other version, using the wife's wage and Pareto-weight.