

ECON3120/4120 Mathematics 2: Exam 2024-12-06

(4 hours. Support material: “Rules and formulas” attachment and Inspera’s simple calculator.)

For the entire problem set:

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier enumerated item (e.g. “(a)”) to solve a later one (e.g. “(c)”), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 Take for granted that the following equation system determines $K = K(p, t)$ and $L = L(p, t)$ as differentiable functions around the point where $(p, t, K, L) = (14, 1, 1, 64)$:

$$\begin{aligned} 3t \cdot (KL)^{1/3} + 2K^{1/2} - pK &= 0 \\ 48t \cdot (KL)^{1/3} + 32L^{1/2} - 7L &= 0 \end{aligned} \tag{S}$$

(a) Differentiate the system.

(b) Approximate $L'_p(14, 1)$.

(Hints: it might be useful to write some of terms on the form $n/16$, and to remember that $64 = 8^2 = 4^3$.)

Problem 2 Consider the matrices $\mathbf{v}_t = \begin{pmatrix} 0 \\ 2 \\ 1 \\ t \end{pmatrix}$, $\mathbf{M}_t = \begin{pmatrix} 0 & 0 \\ 0 & 3 \\ 1 & 0 \\ -4 & 0 \end{pmatrix}$ and $\mathbf{A}_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 \\ t & 0 & -4 & 0 \end{pmatrix}$,

where t is an arbitrary real constant. You may note that \mathbf{v}_t is the first column of \mathbf{A}_t , and that \mathbf{M}_t consists of the third and fourth column of \mathbf{A} .

Parts (a) and (b) will concern the matrix products $(\mathbf{v}_t)^2$, $\mathbf{M}_t' \mathbf{M}_t$, $\mathbf{M}_t \mathbf{A}_t$, $\mathbf{A}_t \mathbf{M}_t$ and $(\mathbf{A}_t)^2$ which may or may not be well-defined (that is a question for you). The prime ($'$) denotes transpose.

(a) For those five products:

i) Decide the order (the «size» $m \times n$) of the matrix product, or clarify that the matrix product is not well-defined.

ii) One or more of the five matrix products will be square but *not* of order 4×4 .

Calculate it (if there are more than one, you choose which one to calculate).

(b) For the well-defined among the five products, *and* also the three matrices \mathbf{v}_t , \mathbf{M}_t and \mathbf{A}_t , in all up to eight:

Calculate its determinant or point out that the determinant is not well-defined.

(c) Consider now for each t the equation system $\mathbf{A}_t \mathbf{x} = \mathbf{v}_t$ (for the unknown \mathbf{x}).

$\mathbf{x} = (1, 0, 0, 0)'$ is one solution. *Is it unique* (i.e. the only solution)?

The answer might depend on t .

Problem 3 Let $f(x, y) = 3x^2 + 4y^2 - xy$, let $h(z) = 5z^2$ and let $F(x, y, z) = f(x, y) + h(z)$.

(a) Show that f , h and F are convex functions. (*Hint*: Sum of ... ?)

Consider the problem

$$\min F(x, y, z) \quad \text{subject to} \quad x + y + z = 6 \quad \text{and} \quad x - z = 2 \quad (\text{P})$$

(b) i) Show that point $(x, y, z) = (3, 2, 1)$ satisfies the associated Lagrange conditions, and
ii) calculate the Lagrange multipliers.

iii) Show that this point is optimal.

(c) Approximate how much the optimal value would change in the following cases:

i) The second constraint is replaced by $x - z = 2 + \frac{1}{100}$

ii) The second constraint is replaced by $x - z = 2 + \frac{1}{100}y$

Problem 4 Let $H > 0$ be a constant. Suppose that the state variable evolves according to $x_{t+1} = x_t - c_t$ and consider the dynamic programming problem

$$v_{t_0}(x_{t_0}) = \min_{c_t} \left\{ c_{t_0}^2 + \frac{1}{2}c_{t_0+1}^2 + \dots + \left(\frac{1}{2}\right)^{T-1-t_0}c_{T-1}^2 + H \cdot \left(\frac{1}{2}\right)^{T-t_0}x_T^2 \right\}$$

- (a) Calculate v_{T-1} (as a function of x_{T-1}).
- (b) Suppose that $T = 2026$ and $t_0 = 2024$. Calculate the optimal c_{2024}^* .

Problem 5 Consider the difference equation (D) and the differential equation (E):

$$x_{t+1} - \frac{3}{4}x_t = 7 - 2^{-t} \quad (\text{D})$$

$$\dot{x} - \frac{3}{4}x = 7 - 2^{-t} \quad (\text{E})$$

(a) For each of the equations (D) and (E):

Find constants q and h such that $q \cdot 2^{-t} + h$ is a particular solution.

(b) Pick either (D) or (E) (your choice):

Find the particular solution x such that $x = 19$ when $t = 0$.

(c) Let r be a real constant. Use *integration by parts* to show that there are constants $\alpha = \alpha_r$, $\beta = \beta_r$, $\gamma = \gamma_r$ such that

$$\int (1-r) \frac{(\ln z)^2}{z^r} dz = \frac{\alpha \cdot (\ln z)^2 + \beta \cdot \ln z + \gamma}{z^{r-1}} + C$$

Note: you are not asked to *calculate* α , β , γ , but beware that one particular r value could need special attention.

(d) Explain how it is possible to calculate $\int (y^2 + 1)^3 \cdot 2y dy$ *without* using the change of variable method («substitution», which is not syllabus).

(End of problem set. Attachment: Rules and formulas.)