

ECON3120/4120 Mathematics 2: exam 2025-12-12

Problems / problem pages to be solved: 1 through 5, *including* this page.

A calculator is available in Inspera.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- Note that even if you are unable to solve a sub-question (e.g. “(a)”), information provided in that sub-questions might possibly be helpful for your answers to later subquestions (e.g. “(c)”).

Suggested weighting :

Each of the letter-enumerated parts («(a)», «(b)» etc.) carries equal weight.

Problem 1 Consider the equation system

$$\begin{aligned}x + z &= 1 \\ax + y + z &= 1 \\x + y + az &= -2\end{aligned}$$

- For what values of the parameter a will the system have exactly one solution?
- Are there any values of a where the system has no solutions?
- Show that for $a = 0$, the inverse of the coefficient matrix \mathbf{A} is

$$\mathbf{A}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

d) Write the following equation system on matrix form. Use the result from part (c) to solve the system.

$$\begin{aligned}x + z &= 1 \\y + z &= 1 \\x + y &= -2\end{aligned}$$

Problem 2 Consider a decision problem over a finite horizon T . A state variable x_t (e.g., an investment balance) evolves according to

$$x_{t+1} = \begin{cases} x_t(1+r) & \text{if } a_t = \text{Continue} \\ 0 & \text{if } a_t = \text{Stop} \end{cases}$$

where a_t is a control variable $a_t \in \{\text{Continue}, \text{Stop}\}$, and where $r > 0$.

The reward $R(x_t, a_t)$ is obtained only when the process is stopped (when the decision maker takes the money out of the investment):

$$R(x_t, a_t) = \begin{cases} 0 & \text{if } a_t = \text{Continue} \\ x_t & \text{if } a_t = \text{Stop} \end{cases}$$

The objective is to maximize

$$V_0(x_0) = \sum_{t=0}^T R(x_t, a_t) \beta^t$$

where $\beta = (1+d)^{-1}$, where $d \geq 0$ is the discount rate.

At the final period T , the process is forced to stop, so the terminal value is $V_T(x_T) = x_T$. You can take for granted that for any t , $V_t(0) = 0$.

a) Explain why the Bellman equation for this problem is:

$$V_t(x_t) = \max \left\{ x_t, \frac{1}{1+d} V_{t+1}(x_t(1+r)) \right\}$$

We assume that $r < d$.

b) Show that if the value function at time $t+1 \leq T$ is given by

$$V_{t+1}(x_{t+1}) = x_{t+1}$$

then $V_t(x_t)$ is also given by

$$V_t(x_t) = x_t$$

and determine the optimal policy (a_t^*) at time t .

Problem 3 Consider the equation system:

$$F_1 : p_1\sqrt{x_1} - 2p_2\sqrt{x_2} = 0$$

$$F_2 : p_1x_1 + p_2x_2 - m = 0$$

Assume that x_1, x_2, p_1, p_2, m are all non-negative. This system implicitly defines the demand functions $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.

a) Differentiate the system

A particular solution to the original system is $x_1 = 9$, $x_2 = 4$, $p_1 = 4$, $p_2 = 3$, and $m = 48$.

b) Use the system from Part (a) to estimate the value of the cross-price derivative $\partial x_1 / \partial p_2$ at this particular point.

Problem 4 Let t and α be constants such that $t, \alpha > 0$. Consider the integral

$$I(t) = \int_0^\alpha x e^{tx} dx. \quad (1)$$

(a) Use integration by parts to show that

$$I(t) = \frac{\alpha e^{t\alpha}}{t} - \frac{1}{t} \int_0^\alpha e^{tx} dx. \quad (2)$$

(b) Set $J(t) = \int_0^\alpha e^{tx} dx$. Show that $J'(t) = I(t)$

Consider the differential equation

$$tI'(t) + I(t) = \alpha^2 e^{t\alpha} - \frac{\alpha e^{t\alpha}}{t} + \frac{e^{t\alpha} - 1}{t^2}. \quad (3)$$

(c) Show that if $\hat{I}(t)$ is a particular solution to the differential equation (3), then $I(t) = \hat{I}(t) + \frac{C}{t}$, with $C \in \mathbb{R}$, also solves (3)

One particular solution to (3) is

$$I(t) = \frac{(\alpha t - 1)e^{\alpha t} + 1}{t^2},$$

(d) Find the particular solution to (3) that satisfies $I(1) = 1$. (Hint: find C)

Problem 5 A household consist of a husband H and a wife W . They share a joint utility function $U(c_H, c_W, l_H, l_W)$ over consumption c and leisure l of the husband and wife, represented by Equation (4) *:

$$U(c_H, c_W, l_H, l_W) = \frac{c_H^{1-\gamma}}{1-\gamma} + \frac{c_W^{1-\gamma}}{1-\gamma} + \frac{1}{1-\phi} [\alpha l_H + (1-\alpha)l_W]^{1-\phi} \quad (4)$$

They have a common budget constraint represented by Equation (5). The parameter κ represents the bargaining power of the husband. All Greek letters are constants between 0 and 1, and wages are positive. The budget constraint is:

$$w_H(1-l_H) + w_W(1-l_W) = \kappa p c_H + (1-\kappa)q c_W \quad (5)$$

with p and q being constant prices of the commodities bought by the husband and wife, respectively.

- (a) Write up the Lagrangian for the utility maximization of this household and state all the associated *Lagrange conditions*.
- (b) Assume an interior solution. Find an expression for the optimal ratio of consumption $\frac{c_H}{c_W}$ in terms of the parameters q, p and κ .
- (c) Assume again an interior solution. Show that the first-order conditions imply:

$$\frac{\alpha}{1-\alpha} = \frac{w_H}{w_W} \quad (6)$$

- (d) Assume that there is an interior solution, and in particular that the condition from (c) holds. Show that this implies:

$$\alpha l_H + (1-\alpha)l_W = \left(-\lambda \frac{w_H}{\alpha}\right)^{-1/\phi} = \left(-\lambda \frac{w_W}{(1-\alpha)}\right)^{-1/\phi} \quad (7)$$

(end of problem set)

*In economic terms: their utility in consumption is *separable* while their utility in leisure is *non-separable*, with a Pareto-weight α on the husband's leisure.