Final Exam
ECON 4310, Fall 2019

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	**Max**
Exercise A		**70**
Exercise B		**50**
Exercise C		**40**
Exercise D		**30**
Σ		**190**

Grade: __________

Exercise A:

A General Equilibrium Model with Population Growth and Taxes (70 points)

Consider an economy in which each household has a utility function given by

$$\log(c_{1t}) + \beta \log(c_{2t+1}).$$

The household works for a wage w_t when young and has no labor income when old, but can save at a gross real interest rate $1 + r_{t+1}$. In each period a new generation is born whose size is $1 + n$ times as large as the previous generation. Thus the size of a generation born at time t is $N_t = (1 + n)^t$. For future reference define the per-capita (of the young generation) capital stock as $k_t = \frac{K_t}{N_t}$.

(a) (10 Points) Derive the optimal consumption and savings choice (c_{1t}, s_t) of the household.

Solution:

$$\begin{aligned} c_{1t} &= \frac{1}{1+\beta} w_t \\ s_t &= \frac{\beta}{1+\beta} w_t \end{aligned}$$

(b) (20 Points) Suppose the wage is given by the marginal product of labor,

$$w_t = (1 - \alpha) (k_t)^\alpha$$

Use the market clearing condition in the capital market,

$$N_t s_t = K_{t+1}$$

to derive how the per capita capital stock k_{t+1} tomorrow depends on the per capita capital stock k_t today.

Solution:

Divide both sides by N_t

$$s_t = \frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = (1 + n) k_{t+1}$$

But since

$$s_t = \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta}(1-\alpha)\,(k_t)^{\alpha}$$

we plug in to obtain

$$\begin{aligned}(1+n)k_{t+1} &= \frac{\beta}{1+\beta}(1-\alpha)\,(k_t)^{\alpha} \\ k_{t+1} &= \frac{\beta}{(1+\beta)(1+n)}(1-\alpha)\,(k_t)^{\alpha}\end{aligned}$$

(c) (10 Points) Find the steady state capital stock(s) in the economy. How many steady states are there?

Solution:

---

there are two steady states, $k = 0$ and the positive k^* that solves

$$k^* = \frac{\beta}{(1+\beta)(1+n)}(1-\alpha)\,(k^*)^{\alpha}$$

or

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

(d) (20 Points) Now suppose the government levies a proportional tax $\tau > 0$ on labor income (wages), so that now income in the first period of a household's life is given by $(1-\tau)w_t$. The government uses the tax receipts to buy tanks that neither yield utility nor affect the production function. How is the steady state capital stock in the economy affected by this tax? Explain.

Solution:

The only thing that changes is that

$$s_t = s_t = \frac{\beta}{1+\beta}(1-\tau)w_t = \frac{\beta}{1+\beta}(1-\tau)(1-\alpha)\,(k_t)^{\alpha}$$

Using the same steps (no need to repeat them) delivers

$$k^* = \left[\frac{\beta(1-\alpha)(1-\tau)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

and thus the capital stock is lower with the tax. The tax reduces disposable income for the households, thus saving and in turn the long run (that is, steady state) capital stock in the economy.

(e) (10 Points) Consider the same situation as in question 4, but now suppose that the government returns the tax receipts to households as lump-sum transfers T_t. Thus income of the household is now

$$(1-\tau)w_t + T_t$$

Transfers in turn are given by tax receipts

$$T_t = \tau w_t.$$

How does your answer to question 4. change? Explain.

Solution:

now total income of the household in period 1 is given by

$$(1-\tau)w_t + T_t = (1-\tau)w_t + \tau w_t = w_t$$

and thus the answer is identical to that in 3, and the steady state is larger now than in question 4. The additional income from the transfer exactly offsets the tax, the households have higher disposable income with the transfers than if the government spends the money on tanks, and thus savings increase, boosting the capital stock in the economy.

Exercise B:

A Three Period Model (50 Points)

Frodo lives for three periods. In the first period he acts in a movie and earns $100,000. In the second period he acts in a movie and earns $150,000 and in the last period of his life he acts in a movie and earns $225,000. His utility function is given by

$$u(c_1) + \frac{2}{3}u(c_2) + \frac{4}{9}u(c_3)$$

We assume, as always, $u'(c) > 0$ and $u''(c) < 0$. The market interest rate is $r = 50\%$ per period. He does not have any bequest from his parents ($A = 0$)

(a) (10 Points) Is Frodo's consumption profile increasing, decreasing or constant with age? Explain your answer?

Solution:

$\beta = 2/3$, $1 + r = 1.5$, thus $\beta(1 + r) = 1$. Consumption is constant over the life cycle.

(b) (10 Points) Solve for the optimal consumption levels c_1, c_2, c_3 explicitly.

Solution:

Answer: the budget constraint reads as

$$c_1 + \frac{c_2}{1.5} + \frac{c_3}{2.25} = 100,000 + \frac{125,000}{1.5} + \frac{225,000}{2.25} = 300,000$$

Since $c_1 = c_2 = c_3$ we have

$$c_1 = c_2 = c_3 = \frac{300,000}{1 + \frac{1}{1.5} + \frac{1}{2.25}} = 142,105$$

(c) (10 Points) Solve for Frodo's savings in the first period of his life sav_1 and Frodo's financial asset position s_1 at the end of period 1. Recall that $s_0 = A = 0$ and that $r = 50\%$.

Solution:

$$sav_1 = y_1 - c_1 = -42,105 = s_1.$$

(d) (20 Points) Now suppose Frodo cannot borrow. Determine the optimal consumption levels c_1, c_2, c_3, his optimal savings choices sav_1, sav_2, sav_3 and his asset positions s_1, s_2, s_3 over Frodo's life cycle.

Solution:

Frodo's optimal consumption is constant, but his income is increasing with his age. Thus the borrowing constraint is binding in each period. Consequently

$$\begin{aligned} c_1 &= 100,000 \\ c_2 &= 150,000 \\ c_3 &= 225,000 \end{aligned}$$

and $sav_1 = sav_2 = sav_3 = 0$, as well as $s_1 = s_2 = s_3 = 0$.

Exercise C:

Taxation and Labor Supply (40 points)

Consider a household that chooses consumption c and labor supply l to solve the following maximization problem

$$\begin{aligned} & \max_{c,l \geq 0} \left\{ \log(c) - \frac{(l)^2}{2} \right\} \\ & s.t. \\ c &= (1-\tau)l + Tr \end{aligned}$$

where τ is the labor income tax rate, Tr is a transfer by the government. Note that I set the wage per hour work to one to simplify your life.

(a) (10 Points) Derive the intratemporal optimality condition, relating labor supply and consumption.

Solution:

Attaching Lagrange multiplier to the budget constraint and taking first order conditions yields

$$\begin{aligned} \frac{1}{c} &= \lambda \\ l &= (1-\tau)\lambda \end{aligned}$$

Combing yields

$$l = \frac{(1-\tau)}{c}$$

(b) (10 Points) Suppose $Tr = 0$. Solve for optimal labor supply.

Solution:

Plugging in for c from the budget constraint yields

$$l = \frac{(1-\tau)}{(1-\tau)l + Tr} \tag{1}$$

and for $Tr = 0$ this yields

$$l = \frac{(1-\tau)}{(1-\tau)l}$$

and thus $l = 1$ (the $l = -1$ solution is not economically reasonable and also violates $l \geq 0$.

(c) (20 Points) Use your answer in 1. to compare optimal labor supply in two scenarios

1. The tax revenues are thrown into the ocean and do not benefit the household at all, that is $Tr = 0$

2. The tax revenues are rebated back to the household, which yields $Tr > 0$.

Under which scenario is labor supply higher, and why? You have to justify your answer even if you do not provide formal calculations.

Solution:

The household works less under scenario b) since the extra income from TR induces a positive income effect on both goods the household likes, consumption c and leisure (the absence of labor). Mathematically one can see this by realizing that in scenario a) we have $l = 1$ and for scenario b) we can rewrite equation (1) as

$$
\begin{aligned}
l^2 &= \frac{(1-\tau)l}{(1-\tau)l + Tr} \\
l^2 &= \frac{1}{1 + \frac{Tr}{(1-\tau)l}} < 1
\end{aligned}
$$

and thus $l < 1$. [It is true there is an l on the right hand side of this equation and thus it is messy to solve for or l explicitly since it results in a quadratic equation, but whatever $l \geq 0$ is on the right hand side, the right hand side is less than one and thus l must be less than 1 under scenario b), confirming economic intuition.]

Exercise D:

Social Security (30 points)

Consider a household that potentially lives for two periods. Let p denote the probability that she survives to the second period. Her utility function is given by

$$\log(c_1) + p\log(c_2)$$

where c_1 is first period consumption and c_2 is second period consumption if the household is alive in the second period. The household has income $y_1 = 10,000$ in the first period of life, but no labor income in the second period of life. Thus the budget constraints read as

$$\begin{aligned} c_1 + s &= (1-\tau)10000 \\ c_2 &= (1+r)s + b \end{aligned}$$

where τ is the social security tax rate and b are social security benefits. The population grows at rate 100% so that there are twice as many young people as there are old people. There is no income growth over time (that is $g = 0$).

(a) (10 Points) Write down the budget constraint of the *government*, relating social security benefits b to taxes τ and the survival probability p.

Solution:

$$pb = \tau 10,000 * 2$$

(b) (10 Points) Suppose the survival probability is $p = 0.8$. What does the social security tax rate τ have to be so that the social security replacement rate is 40%?

Solution:

$$rr = \frac{b}{10,000} = 0.4$$

Thus we need

$$b = 4000$$

From the government budget constraint

$$b = 4000 = \frac{\tau 20,000}{p} = \frac{\tau 20,000}{0.8}$$

Solving this for τ yields

$$\tau = \frac{0.8 * 4,000}{20,000} = 16\%$$

(c) (10 Points) Suppose that still $p = 0.8$ and that the interest rate is $r = 2 = 200\%$. What is the social security tax rate τ and associated benefit b the government should choose to maximize utility of the typical household in this economy. You need to explain your answer.

Solution:

The gross return on social security is $\frac{(1+n)(1+g)}{p} = \frac{2}{0.8} < 3 = 1 + r$ and thus social security is inferior to saving privately. Thus the government should choose $\tau = b = 0$, and the optimal system is no system.