Final Exam Solution
ECON 4310, Fall 2020

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	**Max**
Exercise A		**40**
Exercise B		**40**
Exercise C		**60**
Σ		**140**

Grade: __________

Exercise A:

Taxation and Labor Supply (40 points)

Consider a household that chooses consumption c and labor supply l to solve the following maximization problem

$$\max_{c,l\geq 0} u(c,l)$$
$$\text{s.t.}$$
$$c = (1-\tau)l,$$

where τ is the labor income tax rate. Note that the wage per hour work is set to one.

(a) (10 Points) Consider the following utility function $u(c,l) = \log(c) - l^2/2$. Derive the optimality condition, relating labor supply and consumption. Solve for labor supply l.
ANS: FOC:

$$(1-\tau)\frac{1}{c} = l,$$

substitute into BC

$$c = (1-\tau)(1-\tau)\frac{1}{c} \Rightarrow$$
$$c = (1-\tau),$$
$$l = 1.$$

(b) (10 Points) How does labor respond to labor tax increase? Explain your answer by analyzing the income and the substitution effects of labor supply.
ANS: no response. income and substitution effects cancel each other given log utility.

(c) (10 Points) Now consider an alternative utility function $u(c,l) = \log\left(c - l^2/2\right)$. Derive the optimality condition, relating labor supply and consumption. Solve for labor supply l.
ANS: FOC:

$$(1-\tau)\frac{1}{c-l^2/2} = \frac{1}{c-l^2/2}l \Rightarrow$$
$$l = (1-\tau).$$

(d) (10 Points) How does labor respond to a labor tax increase with the alternative utility function? Explain your answer by analyzing the income and the substitution effects of labor supply.
ANS: labor decreases if tax increases because of the substitution effect. No income effect.

Exercise B:

Ricardian Equivalence (40 Points)

Consider an 2-period OLG model where one period is 30 years and population growth is 0. Suppose that at the beginning of period t, the government announces a special new expenditure plan for that period: spending g per capita to fight the global warming. The government has come up with three policies to finance the expenditure:

1. Finance the expenditure with immediate lumpsum taxes on all households living in period t.
2. Issue government debt, and repay that debt, including interest, in the next period using lumpsum taxes on all households living in period $t+1$.
3. Issue government debt and simply pay the interest on that government debt forever, without ever redeeming the debt itself.

Assume that the interest rate is $r = 100\%$ (so that $1 + r = 2$).

Now consider a household who lives in period t and $t+1$, i.e., it is young in period t and old in period $t+1$. It works when it is young and its labor income is w_t. When it is old, it does not work and has no labor income. it has utility function

$$\log(c_t^y) + \beta \log(c_{t+1}^o),$$

where $\beta = 1/\,(1+r) = 1/2$, and c_t^y and c_{t+1}^o denote its consumption in period t when it is young and consumption in period $t+1$ when it is old, respectively.

(a) (10 Points) Assume that the household can borrow and lend as much as it wants. How does it rank those three tax policies? Explain.
ANS: 3>2=1. In 3, most tax burdens are on future generations. 2 and 1 are the same, given zero population growth.

(b) (10 points) Now suppose that the household cannot borrow any longer and $g = w_t/4$. How does it rank those three tax policies? Explain.
ANS: 3>2=1. In 3, most tax burdens are on future generations. 2 is the same as 1, as its savings are high enough.

(c) (10 points) Now consider a household from the next generation, who lives in period $t+1$ and $t+2$, and it can borrow and lend freely. How does it rank those tax policies? Explain.
ANS: 1>3>2. In 1 it pays nothing; in 3, it shares the tax burden will all future generations; in 2, it shares half of the tax burden with generation t.

(d) (10 points) Finally, suppose that all households are the same and can live forever instead of just 2 periods. How does the representative household rank those tax policies and how does its ranking depend on how much it can borrow? Explain.

(Hint: you just need to generally and qualitatively discuss the relation between the ranking and the borrowing constraint and you don't need to solve for any expression or number.)

ANS: No borrowing constraint: $3 = 2 = 1$. With borrowing constraint: 3>=2>=1. Better to delay tax payments.

Exercise C:

A Real Business Cycle Model (60 Points)

Consider a representative household of a closed economy. The household has a planning horizon of two periods and is endowed with the following preferences over consumption, c,

$$U = u(c_1) + \beta \mathrm{E} u(c_2(s_2)),$$

where $\beta = 1/2$, and $u(c) = \log c$ (natural log utility function).

The variable s_2 denotes the state of the economy in the second period which follows the stochastic process

$$s_2 = \begin{cases} s_G, \text{ with prob. } p \\ s_B, \text{ with prob. } 1-p, \end{cases}$$

where $p = 0.5$ and the household conditions the consumption, $c_2(s_2)$, in the second period on the state, s_2. Assume the household's labor supply is exogenous and always equal to 1.

Labor market assumptions:

Assume that in each period and in each state of the economy, s_t, there is a linear (in labor n_t) production technology of the form

$$y_t(s_t) = A_t(s_t) n_t(s_t),$$

and the labor market is assumed to be competitive. Assume the labor productivity in the first period is given by $A_1 = A = 1$, and the labor productivity is higher in the good state of the second period,

$$A_2(s_G) = 1 + \epsilon > A_2(s_B) = 1 - \epsilon, \quad \epsilon = \frac{2}{3}$$

than in the bad state of the second period. The wages are denoted as w_1, $w_2(s_G)$, and $w_2(s_B)$.

Asset market assumptions:

Assume the household does have access to a risk-free asset, a_2, and the associated interest rate is denoted as $r_2 = 100\%$.

We can denote a variable x in period 2 in stage G or B simply as x_{2G}. For example, $A_2(s_G)$ as A_{2G}, $w_2(s_B)$ as w_{2B}, $c_2\left(s_G\right)$ as c_{2G} and $\lambda_2\left(s_G\right)$ as λ_{2G}, etc.

(a) (5 Points) Find the equilibrium wages, w_1, w_{2G}, and w_{2B}. (In the final expressions, substitute the numerical values into the variables whenever possible, e.g., substitute $1+\epsilon$ for A_{2G} and 1 for r. Same for the questions below.)
ANS: Since labor markets are competitive, we should have wages equal to productivity as follows:

$$\begin{aligned} w_1 &= A = 1, \\ w_{2G} &= A_{2G} = 1+\epsilon, \\ w_{2B} &= A_{2B} = 1-\epsilon. \end{aligned}$$

(b) (5 Points) Write down the state-by-state budget constraints for the household.
ANS: the state-by-state budget constraints for the household are:

$$\begin{aligned} c_1 + a_2 &= 1, \\ c_{2G} &= 1+\epsilon+2a_2, \\ c_{2B} &= 1-\epsilon+2a_2. \end{aligned}$$

(c) (10 Points) Let $(\lambda_1, \lambda_{2G}, \lambda_{2B})$ denote the Lagrange multipliers of the state-by-state budget constraints. State the representative agent's Lagrangian. (Note that the expected utility for the second period is the summation of utility across good and bad states, weighted by probability, i.e., $\mathrm{E}u(c_2) = pu(c_{2G}) + (1-p)u(c_{2B})$. The values of the Lagrange multipliers are not known yet, so you don't need to find their values at this stage)
ANS: The Lagrangian can be written in the state-ordered form as

$$\begin{aligned} \mathcal{L} = {} & u(c_1) + \lambda_1 \left[w_1 - a_2 - c_1\right] \\ & + \beta p \left[u(c_{2G})\right] + \lambda_{2G} \left[w_{2G} + 2a_2 - c_{2G})\right] \\ & + \beta(1-p) \left[u(c_{2B})\right] + \lambda_{2B} \left[w_{2B} + 2a_2 - c_{2B}\right]. \end{aligned}$$

(d) (10 Points) Derive the optimality conditions with respect to consumption and savings, i.e., $c_1, c_2(s_G), c_2(s_B)$ and a_2 by using multipliers.
ANS: The optimality conditions with respect to the choices are:

$$0 = \frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda_1 \tag{1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial c_2(s_G)} = \beta p u'(c_2(s_G)) - \lambda_2(s_G) \tag{2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial c_2(s_B)} = \beta(1-p) u'(c_2(s_B)) - \lambda_2(s_B) \tag{3}$$

$$0 = \frac{\partial \mathcal{L}}{\partial a_2} = -\lambda_1 + \left[\lambda_2(s_G) + \lambda_2(s_B)\right](1+r_2). \tag{4}$$

(e) (10 Points) Derive the stochastic consumption Euler equation (only involves c_1,c_{2G},c_{2B}, β and r_2 and no multipliers).
ANS: The stochastic consumption Euler equation is given by:

$$u'(c_1) = \beta \mathbf{E}\left[u'(c_2(s_2))\right](1+r_2). \tag{5}$$

(f) (10 Points) Solve for optimal consumption, savings. (Now you need to find the numerical values for c_1, c_{2G}, c_{2B} and a_2. Hint: the euler equation can be transformed into a quadratic equation of a_2, and the positive root is the optimal a_2).
ANS:

$$\begin{aligned}
\frac{1}{1-a} &= \beta(1+r_2)\left(p\frac{1}{2a+1+\epsilon} + (1-p)\frac{1}{2a+1-\epsilon}\right) \Rightarrow \\
\frac{1}{1-a} &= 1 * \left(\frac{1}{2}\frac{1}{2a+1+\epsilon} + \frac{1}{2}\frac{1}{2a+1-\epsilon}\right) \Rightarrow \\
6a^2 + 3a - \epsilon^2 = 0 \Rightarrow & \\
a &= \frac{-3 \pm \sqrt{9+24\epsilon}}{12} \\
&= \frac{-3 \pm 5}{12}.
\end{aligned}$$

$a = 1/6 > 0$ is the solution.

(g) (10 Points) If the households cannot save, what are the consumption levels c_1,c_{2G}, and c_{2B}? Is the utility higher or lower than that with the risk-free asset a_2? What is the interest rate for the risk-free asset that makes the household optimally choose not to save?
ANS:

$$\begin{aligned}
c_1 &= w_1 \\
c_2(s_2) &= w(s_2), \ \forall s_2 \in S,
\end{aligned}$$

or,

$$\begin{aligned}
c_1 &= w_1 = A, \\
c_2(s_G) &= w_2(s_G) = A_2(s_G) > A, \\
c_2(s_B) &= w_2(s_B) = A_2(s_B) < A.
\end{aligned}$$

Lower utility without consumption smoothing.