Final Exam
ECON 4310, Fall 2021

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	**Max**
Exercise A		**100**
Exercise B		**70**
Exercise C		**40**
Σ		**210**

Grade: __________

Exercise A:

Ramsey Growth Model (100 points)

Consider a discrete-time version of Ramsey's growth model. The economy is closed and we consider a representative agent with the following preferences over consumption

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where c_t denotes period t consumption and $\beta \in (0, 1)$ is the subjective discount factor. The momentary utility function is of the form

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1 - \theta},$$

with $\theta > 1$. Every period the agent earns a wage w_t (the labor supply is exogenously set to 1 unit), an interest $r_t a_t$ from her assets holdings and she is subject to the lump-sum tax τ_t. In equilibrium, the agent will choose the sequence consumption and asset holdings $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize U subject to the period-by-period budget constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t, \tag{2}$$

for a given a_0. The agent is atomic and her decisions do not influence aggregate variables, thus she takes the sequence of taxes, wage rates and interest rates as given.

The representative firm demands physical capital k_t and labor n_t to produce output y_t with the Cobb-Douglas technology

$$y_t = k_t^{\alpha} n_t^{1-\alpha}. \tag{3}$$

The firm is atomic and acts as a price-taking profit maximizer. Capital can be rented at the rental rate $R_t = r_t + \delta$ (note that the depreciation rate δ is the difference between the rental rate and the interest rate) while labor costs w_t. There is no government expenditure.

The first welfare theorem applies to this economy such that the competitive equilibrium is efficient in the Pareto sense. Thus, we know that the solution to the social planner's problem (which characterizes the Pareto efficient allocation) is equivalent to the competitive market equilibrium. According to the social planner's solution, the same consumption Euler equation and resource constraint (goods market clearing) along with the so-called transversality condition (which stands in for the no-Ponzi condition)

$$\frac{c_{t+1}}{c_t} = [\beta(1 + r_{t+1})]^{1/\theta} = \left[\beta(1 + \alpha k_{t+1}^{\alpha-1} - \delta)\right]^{1/\theta}$$

$$k_{t+1} - k_t = k_t^{\alpha} - \delta k_t - c_t$$

$$\lim_{t\to\infty} \beta^t c_t^{-\theta} k_{t+1} = 0$$

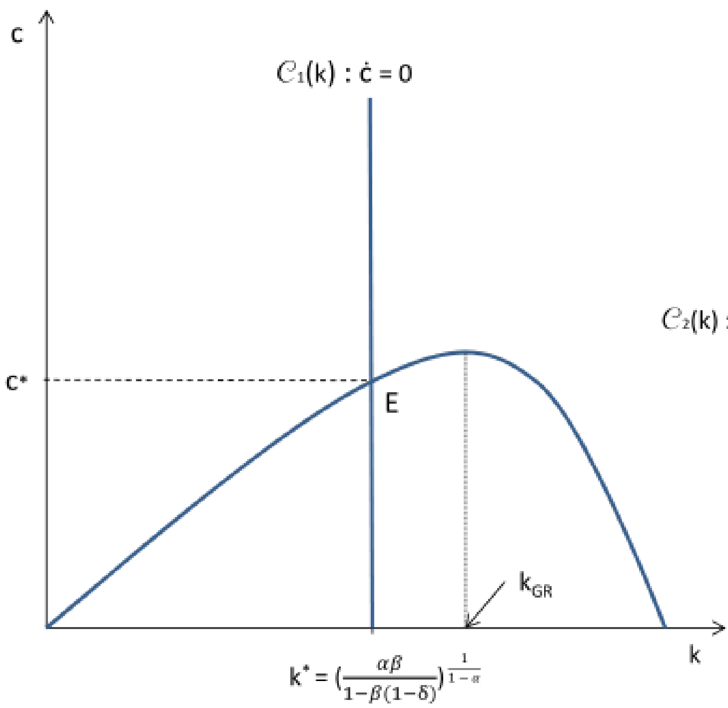
determine the optimal solution of the dynamic system. We can define two correspondances: One characterizes all possible combinations of (c_t, k_t) when consumption is constant,

$$\mathcal{C}_1(k) \equiv \left\{ c \in [0,\infty) : c_{t+1}/c_t = \left[\beta(1+\alpha k^{\alpha-1}-\delta)\right]^{1/\theta}, c_{t+1} = c_t = c \right\},$$

and one captures all combinations if the physical capital stock is constant,

$$\mathcal{C}_2(k) \equiv \{c \in [0,\infty) : c = k_t^{\alpha} - (k_{t+1} - (1-\delta)k_t), k_{t+1} = k_t = k\}.$$

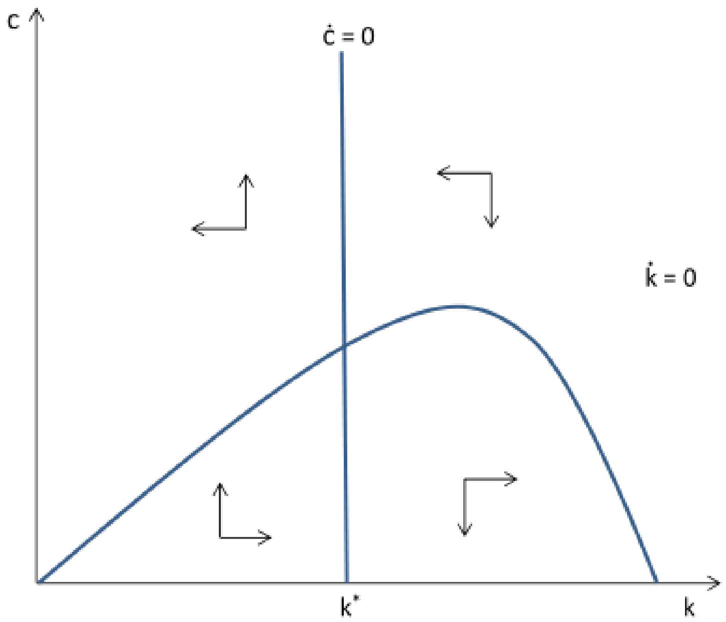
(a) (15 points) Draw the two correspondances, $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$, in a diagram with k on the horizontal axis and c on the vertical axis, the so called phase diagram.



ANS:

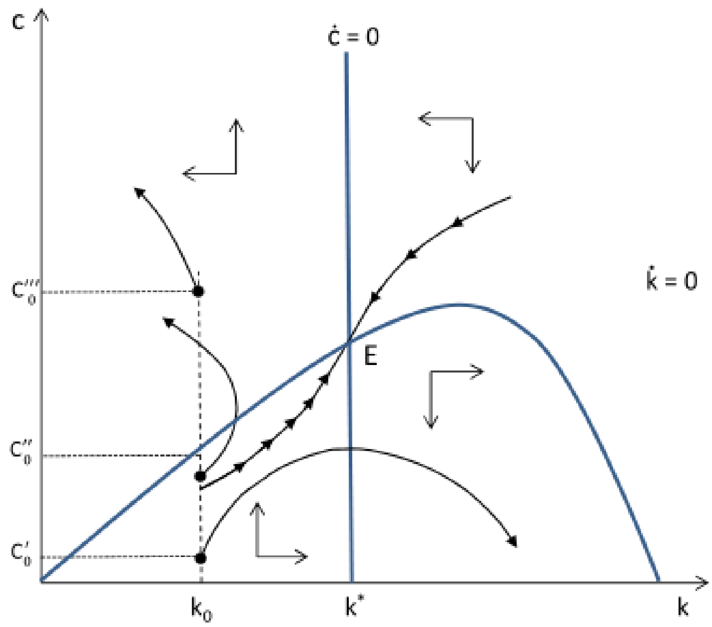
(b) (10 points) Comment on the unique point in the phase diagram where the two correspondances intersect.
ANS: By the construction of $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$ the intersection corresponds to the resting point of the dynamic system, the so called steady state where both consumption c_t and physical capital k_t remain constant over time.

(c) (15 points) Using the phase diagram, illustrate in what direction (c_t, k_t) will move (in all areas of the (c, k)-space).



ANS:

(d) (15 points) Sketch (we do not know the precise shape at this stage) the saddle path leading to the steady state. Explain why any initial consumption off the saddle path cannot be an equilibrium.

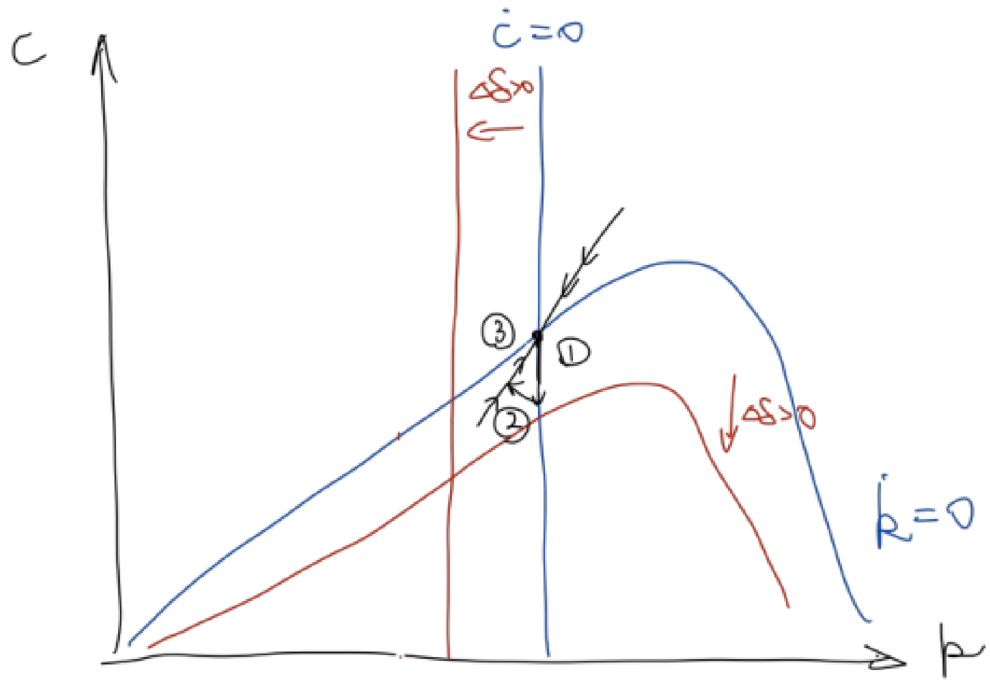


ANS:

Assume that the economy is in a stationary equilibrium in period t_0.

(e) (15 points) Consider an unexpected and temporary increase of $\Delta\delta > 0$ in depreciation rate from period t_0 until period $t_1 = t_0 + 1$. Sketch the dynamics of consumption and physical capital back to the steady-state in the phase diagram. Sketch the time path of the wage rate and interest rates.

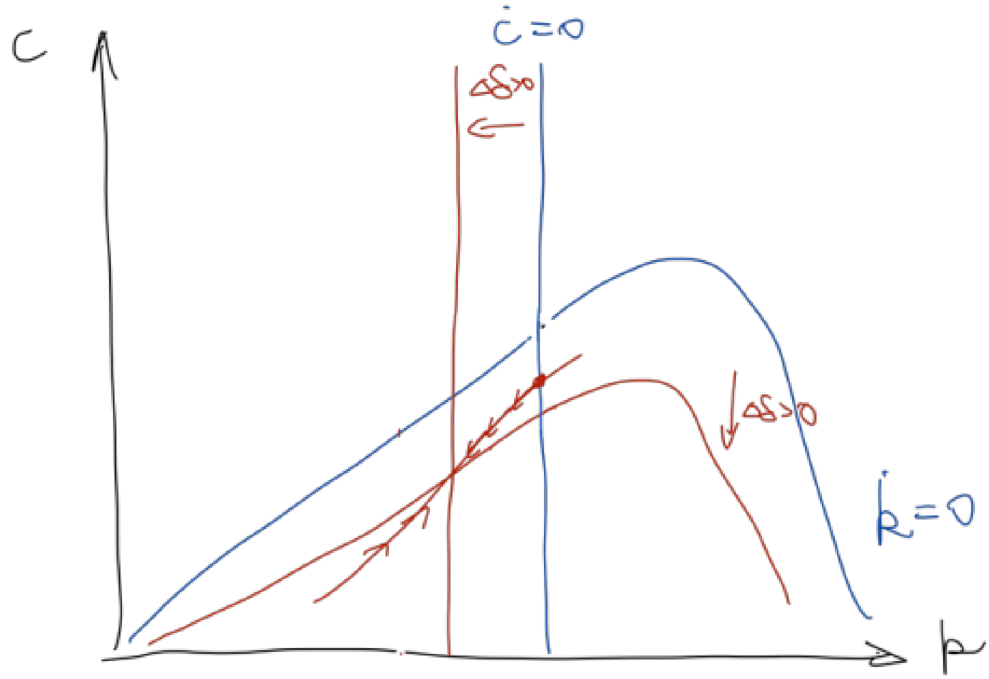
ANS: Unexpected increase of depreciation implies lower level of resource at period t_0 for consumption c_{t_0} and capital k_{t_0+1}, so c_{t_0} drops (as arrow (1) in the figure shows) and k_{t_0+1} also drops . From period $t_1 = t_0 + 1$ on, as depreciation rate jumps back to the initial level, the dynamics of c_t and k_t follows the initial equilibrium path starting from $k_{t_0+1} < k^*$, i.e., c_t and k_t increase and converge to the initial steady state levels following the saddle path. As arrow (2) in the figure shows, c_{t_0+1} is higher than c_{t_0}, because k_{t_0+1} is lower than k^* and increase rate r_{t_0+1} is higher than r^*. Arrow (3) in the figure shows how c and k converge to the steady state following the saddle path.
At period t_0, wage rate is the same as the initial steady state level, but the interest rate drops to a lower level as the depreciation rate jumps up. From period t_1 on, as capital gradually increases towards to the initial steady state level, wage rate gradually increases back to the initial steady-state level. Interest rate at period t_1 is higher than the steady state level and then gradually decreases to the steady state level.



(f) (15 points) Consider an unexpected and temporary increase of $\Delta\delta > 0$ in depreciation rate from period t_0 until period $t_1 = t_0 + T, T > 1$. Sketch the dynamics of consumption and physical capital back to the steady-state in the phase diagram. Sketch the time path of the wage rate and interest rates.
ANS: The dynamics is similar to that in part (e) with some modifications. The difference comparing to part (e) is that, arrow 2 takes T periods, instead of 1 period in part (e), to reach the unique saddle path.
During this time both capital and consumption fall. Similarly, wage rate follows the trend of capital: from t_0 to t_1, wage rate decreases for T period, and then from t_1 on, increases towards the initial steady state level. Interest rate drops to a low

level at t_0 because of the jump of depreciation rate, then gradually increases until t_1, as capital gradually increases; at t_1, interest rate jumps to a higher level because the depreciation rate drops back to the initial level, and after t_1, interest rate gradually decreases towards the initial steady state level.

(g) (15 points) Consider an unexpected and permanent increase of $\Delta\delta > 0$ in depreciation rate for all future periods $t \geq t_0$. Sketch the dynamics of consumption and physical capital to the new steady-state in the phase diagram. Sketch the time path of the wage rate and interest rates.
ANS: The dynamics of c and k immediatelly follows the equilibrium path characterized by the dynamic equations with the new depreciation rate $\delta + \triangle\delta$: c_{t_0} drops to a lower level than the initial steady state level, and from $t_0 + 1$ on, c_t and k_t gradually decrease to the new steady state levels.



Exercise B:

A Social Security (70 points)

Consider a household that potentially lives for two periods. Let p denote the probability that she survives to the second period. Her utility function is given by

$$2\sqrt{c_1} + p2\sqrt{c_2}$$

where c_1 is first period consumption and c_2 is second period consumption if the household is alive in the second period. The household has income $y_1 = 60,000$ in the first period of life, but no labor income in the second period of life. Thus the budget constraints read as

$$\begin{aligned} c_1 + s &= 60,000 \\ c_2 &= (1+r)s \end{aligned}$$

(a) (10 Points) Solve the household's maximization problem for optimal consumption and savings, as functions of the interest rate r and the probability of survival p, that is, determine $c_1(r,p), c_2(r,p), s(r,p)$.

Solution:

the intertemporal budget constraint reads as

$$c_1 + \frac{c_2}{1+r} = 60,000 \tag{4}$$

and the optimal consumption choices are

$$\begin{aligned} c_1(r,p) &= \frac{60,000}{1+p^2(1+r)} \\ c_2(r,p) &= \frac{60,000p^2(1+r)^2}{1+p^2(1+r)} \\ s(r,p) &= \frac{60,000p^2(1+r)}{1+p^2(1+r)} \end{aligned}$$

(b) (20 Points) What happens to $c_1(r,p)$ and $c_2(r,p)$ when the gross real interest rate $(1+r)$ increases. Relate your answer to the income, substitution and human wealth effects.

Solution:

$$\frac{\partial c_1(r,p)}{\partial(1+r)} = \frac{-p^2 60,000}{(1+p^2(1+r))^2} < 0$$
$$\frac{\partial c_2(r,p)}{\partial(1+r)} = \frac{120000p^2(1+r)(1+p^2(1+r)) - 60000p^4(1+r)^2}{(1+p^2(1+r))^2} > 0$$

The human wealth effect is here 0 because there is no income to be discounted in the 2nd period of life. For $c_1(r,p)$ the income effect from a change in $(1+r)$ is positive and the substitution effect is negative (one would substitute towards consumption in period 2 because $c_2(r,p)$ becomes relatively cheaper). Since $\frac{\partial c_1(r,p)}{\partial(1+r)} < 0$ the substitution effect dominates. For $c_2(r,p)$ the income and substitution effects are both positive and thus $\frac{\partial c_2(r,p)}{\partial(1+r)} > 0$.

(c) (10 Points) Suppose that $p = 0$. What are the optimal consumption and savings choices? Explain.

Solution:

from above we have

$$c_1 = 60,000$$
$$c_2 = s = 0$$

Obviously, a person that knows for sure that she is going to die after the second period will consume all her income in the first period and not save anything.

(d) (10 Points) Now the government introduces a social security system of the following form. In the first period, everybody pays a payroll tax rate of $\tau = 10\%$. In the second period those that are alive get social security benefits b. The budget constraint of the government reads as

$$pb = \tau 60,000 = 6000$$

so that

$$b = \frac{6000}{p}$$

Repeat question (a)

Solution:

the budget constraints become

$$\begin{aligned} c_1 + s &= 54,000 \\ c_2 &= (1+r)s + \frac{6000}{p} \end{aligned}$$

Consolidating yields the lifetime budget constraint

$$c_1 + \frac{c_2}{1+r} = 54,000 + \frac{6000}{p(1+r)} = I(p,r) \tag{5}$$

Maximizing utility subject to this constraint yields

$$\begin{aligned} c_1^{SS}(r,p) &= \frac{I(p,r)}{1+p^2(1+r)} \\ c_2^{SS}(r,p) &= \frac{I(p,r)p^2(1+r)^2}{1+p^2(1+r)} \\ s^{SS}(r,p) &= \frac{I(p,r)p^2(1+r)}{1+p^2(1+r)} \end{aligned}$$

(e) (10 Points) Suppose that $p = 0.1$ and $r = 0.3$. Would you rather live in a world with or without social security?

Solution:

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see part (f)

(f) (10 Points) Give a general condition on (r,p) such that the household strictly prefers to live in a world with, rather than without social security.

Solution:

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the social security system only affects the intertemporal budget constraint. Comparing (4) and (5) we see that social security is preferred if and only if

$$\frac{1}{p(1+r)} > 1$$

or $p(1+r) < 1$, that is, if the interest rate is sufficiently low and the survival probability is sufficiently low. Clearly if $p = 0.1$ and $r = 0.3$, then $p(1+r) = 0.13 < 1$, and thus the household prefers to live with, rather than without social security.

Exercise C:

The Solow Model (40 points)

Consider an economy that is described by the Solow growth model. The output is produced with the following production function: $Y_t = K_t^{\frac{1}{3}}(A_tL_t)^{\frac{2}{3}}$, where K_t is the amount of capital, and L_t is the amount of labour used in production in period t. A_t is the labour-augmenting productivity that grows at rate $g = 0.02$, while the population grows at rate $n = 0.01$. Capital depreciation rate is equal to $\delta = 0.1$.

(a) (10 Points) Suppose the savings rate is equal to s. Find the steady state in this economy. What happens to the capital per capita, output per capita and consumption per capita in the steady state?

Solution:

First, express the production function in terms of effective workers:

$$y_t = \frac{Y_t}{A_tL_t} = \left(\frac{K_t}{A_tL_t}\right)^{\frac{1}{3}} = k_t^{\frac{1}{3}}$$

In the steady state, we have:

$$sy^* = (n + g + \delta)k^*$$

or

$$s(k^*)^{\frac{1}{3}} = (n + g + \delta)k^* \quad \Rightarrow \quad k^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{3}{2}}$$

In the steady state, capital per capita, output per capita and consumption per capita all grow at the rate $g = 0.02$.

(b) (10 Points) Compute the steady state wage rate, w_t, and the rental rate of capital, r_t, (the interest rate of this economy will be $r_t - \delta$) in this economy.

Solution:

The wage rate is given by the marginal product of labor in the production function

$$w_t = \frac{\partial Y_t}{\partial L} = \frac{2}{3}K_t^{\frac{1}{3}}(A_tL)^{-\alpha}A_t = \frac{2}{3}A_tk_t^{\frac{1}{3}} \tag{6}$$

and the rental rate is given by the marginal product of capital in the production function

$$r_t = \frac{\partial Y_t}{\partial K_t} = \frac{1}{3}K_t^{-\frac{2}{3}}(A_tL)^{\frac{2}{3}} = \frac{1}{3}k_t^{-\frac{2}{3}}. \tag{7}$$

(c) (10 Points) Find the golden rule level of capital per effective worker. Find the savings rate that would lead to this level of capital per effective worker in the steady state, s_g.

Solution:

At the golden rule, we must have:

$$MP_k = n + g + \delta$$

or

$$\frac{1}{3}k_g^{-\frac{2}{3}} = n + g + \delta \quad \Rightarrow \quad k_g = \left(\frac{\frac{1}{3}}{n+g+\delta}\right)^{\frac{3}{2}}$$

By comparing the formulas for k^* *and* k_g*, it is clear that we need* $s_g = \frac{1}{3}$.

(d) (10 Points) Suppose that the economy starts in the steady state that corresponds to $s = s_g$, but a large natural disaster distroys 10% of the total capital stock. What will happen in the short and the long run?

Solution:

Output initially falls. Since capital falls below its steady state level, the disaster will be followed by a period when capital and output will grow faster than they were in the steady state. The wage rate initially falls and will for some time grow faster than in the steady state. The interest rate initially increases and gradually declines towards the steady state level. In the long run, the economy will return to the same steady state behavior as before the change.