Final Exam
ECON 4310, Fall 2022

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	**Max**
Exercise A		**100**
Exercise B		**100**
Σ		**200**

Grade: __________

Exercise A:

Ramsey Growth Model (100 points)

Consider a consumer who lives for only two periods denoted by $t = 1, 2$. The consumer is born in period 1 without any financial assets. The consumer's labor income is $w_t \geq 0$ in each of the two periods and her preferences over consumption can be represented by the utility function

$$U(c_1, c_2) = u(c_1) + \beta u(c_2), \tag{1}$$

where the momentary utility function is given by

$$u(c) = \log(c)$$

The consumer can borrow and lend consumption across periods at the given real interest rate, r.

(a) Write down the consumer's net present value budget constraint, and find the optimal consumption and savings over the life-cycle.
Solution:

The agent's budget constraints in the two periods read (implicitly assuming the terminal condition that savings in the second period must be equal to zero)

$$c_1 + s = w_1$$
$$c_2 = (1+r)s + w_2 \quad \Leftrightarrow \quad s = \frac{c_2 - w_2}{1+r}.$$

Substituting out the savings, s, yields the net present value private budget constraint

$$c_1 + \frac{c_2}{1+r} = w_1 + \frac{w_2}{1+r}. \tag{2}$$

Equation (2) formalizes that consumption can be shifted across periods but is bounded by the consumers discounted lifetime income. Maximizing $U(c_1, c_2)$ subject to the lifetime budget constraint in Equation (2) yields the first-order optimality conditions for consumption (let λ denote the Lagrange multiplier on the lifetime budget constraint)

$$0 = u'(c_1) - \lambda$$
$$0 = \beta u'(c_2) - \lambda/(1+r).$$

Note that $\theta = 1$ in the more general derivations. Combining the two yields the consumption Euler equation

$$c_2/c_1 = [\beta(1+r)]^{1/\theta}. \tag{3}$$

Combining Equations (2) and (4) yields

$$c_1 + \frac{[\beta(1+r)]^{1/\theta}}{1+r}c_1 = \frac{1+r+[\beta(1+r)]^{1/\theta}}{1+r}c_1 = w_1 + \frac{w_2}{1+r},$$

which can be reformulated as first period consumption

$$c_1 = \frac{1+r}{(1+r)+[\beta(1+r)]^{1/\theta}}\left(w_1 + \frac{w_2}{1+r}\right) = \frac{1}{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}\left(w_1 + \frac{w_2}{1+r}\right),$$

implying that future consumption must be be

$$c_2 = \frac{[\beta(1+r)]^{1/\theta}}{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}\left(w_1 + \frac{w_2}{1+r}\right).$$

The optimal savings follow immediately

$$\begin{aligned} s &= w_1 - c_1 \\ &= w_1\left(\frac{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}{1+\beta^{1/\theta}(1+r)^{1/\theta-1}} - \frac{1}{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}\right) - \frac{1}{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}\frac{w_2}{1+r} \\ &= \frac{1}{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}\left(\beta^{1/\theta}(1+r)^{1/\theta-1}w_1 - \frac{w_2}{1+r}\right). \end{aligned}$$

(b) Write down the Lagrangean for the consumer problem (not using the present value budget constraint). Derive the optimality condition and compare to the solution in a)

Solution:

Yields the same consumption Euler equation

$$c_2/c_1 = [\beta(1+r)]^{1/\theta}. \tag{4}$$

(c) Derive and discuss the effect of an increase in the gross real interest rate $1+r$

Solution:

Write first period consumption as

$$c_1 = \frac{1}{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}\left(w_1 + \frac{w_2}{1+r}\right),$$

so the first factor only contains one term involving the real interest rate. The derivative with respect to $1+r$ then yields

$$\begin{aligned} \partial c_1/\partial(1+r) = &-\frac{(1/\theta-1)\beta^{1/\theta}(1+r)^{1/\theta-2}}{\left[1+\beta^{1/\theta}(1+r)^{1/\theta-1}\right]^2}\left(w_1 + \frac{w_2}{1+r}\right) \\ &- \frac{1}{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}\frac{w_2}{(1+r)^2}. \end{aligned}$$

(d) Assume that $w_1 = w_2$, $\beta = 1$, $r = 0$. Derive c_1 and c_2.

Solution:

$c_1 = w_1$, $c_2 = w_2$.

(e) Assume that $w_2 = 2 * w_1$, $w_1 = 10$, $\beta = 1$, $r = 0$. Derive c_1 and c_2.

Solution:

$c_1 = 3 * w_1/2 = 15$, $c_2 = 3 * w_1/2 = 15$.

(e) Assume that $w_2 = 2 * w_1$, $w_1 = 10$, $\beta = 1$, $1 + r = 2$. Derive c_1 and c_2.

Solution:

$c_1 = w_1 = 10$, $c_2 = w_2 = 2 * w_1 = 20$.

Exercise B:

Solow Model and climate change (100 points)

Consider a closed economy with a neoclassical production function, exogenous technological progress, A_t, a fixed saving rate, s, and a constant labor force, L, as described by the following equations (the Solow model):

$$K_{t+1} - K_t = sY_t - \delta K_t \tag{5}$$
$$Y_t = K_t^\alpha (A_t L)^{1-\alpha},\ 0 < \alpha < 1, \tag{6}$$
$$A_{t+1} = (1+g)A_t,\ A_0 > 0,$$

where $0 \leq \delta \leq 1$ is the depreciation rate of physical capital

(a) Solve for the steady state capital per efficiency unit labor in this economy.

Solution:

De-trend output and the capital transition equation

$$y_t = \frac{Y_t}{A_t L} = K_t^\alpha (A_t L)^{-\alpha} = k_t^\alpha.$$
$$\frac{K_{t+1}}{A_t L} - k_t = sy_t - \delta k_t.$$
$$k_{t+1} - k_t = sk_t^\alpha - \delta k_t - gk_{t+1}.$$

In steady state $k' = k = k^\star$:

$$0 = s(k^\star)^\alpha - (\delta + g)k^\star$$
$$k^\star = \left(\frac{s}{\delta + g}\right)^{1/(1-\alpha)}$$

(b) What is the competitive wage paid to each worker in the steady state? What is the rental rate of capital in the steady state?

Solution:

$$w_t = \frac{\partial Y_t}{\partial L} = (1-\alpha) K_t^\alpha (A_t L)^{-\alpha} A_t = (1-\alpha) A_t k_t^\alpha$$

$$r_t = \frac{\partial Y_t}{\partial K_t} = \alpha K_t^{\alpha-1} (A_t L)^{1-\alpha} = \alpha k_t^{\alpha-1}.$$

(c) Explain what is meant by the Golden Rule savings rate. Find s_{GR} and state the capital per efficiency unit labor that it implies, k_{GR}.

Solution:

The Golden Rule savings rate, s_{GR}, is the savings rate that is associated with the steady state where consumption is maximized.

$$\max_k (1-s)y \quad \text{s.t.} \quad sy = (\delta + g)k$$
$$\alpha k^{\alpha-1} = (\delta + g)$$
$$k_{GR} = (\frac{\alpha}{\delta + g})^{(\frac{1}{1-\alpha})}$$

Comparing $k^\star$ and k_{GR}, we can see that:

$$s_{GR} = \alpha$$

(d) A large tropical cyclone hits the steady state economy, destroying a significant portion of installed capital at $t = 0$. What are the impacts on output, the wage and rental rate, and the savings rate in that period? Explain.

Solution:

When the level of capital falls output falls, because there is less capital being used in production. In addition, the marginal product of labor falls. Since workers are paid their marginal productivity, their wages fall. On the other hand, when capital is destroyed, the marginal productivity of capital rises, which implies that the rental rate rises. The savings rate is unchanged in the Solow model.

(e) After the storm has passed, what is the rate of change of the capital per efficiency unit? That is, find $\frac{k_{t+1}}{k_t}$. Show that capital is accumulating, i.e. that $\frac{k_{t+1}}{k_t} > 1$.

Solution:

Starting with the capital transition equation:

$$k_{t+1} - k_t = sk_t^\alpha - \delta k_t - gk_{t+1}$$
$$k_{t+1} = \frac{1}{1+g}(sk_t^\alpha + (1-\delta)k_t)$$
$$\frac{k_{t+1}}{k_t} = \frac{1}{1+g}(sk_t^{\alpha-1} + (1-\delta))$$

This is equivalent to showing that capital is accumulating when it falls below the long run equilibrium. Since capital is destroyed due to a temporary shock, it must re-accumulate under the Solow model.

$$\frac{1}{1+g}(sk_t^{\alpha-1} + (1-\delta)) > 1$$
$$k_t^{\alpha-1} > \frac{1+g-(1-\delta)}{s}$$
$$k_t < \left(\frac{s}{\delta+g}\right)^{1/(1-\alpha)}$$
$$k_t < k^\star$$

(f) Now suppose that the same economy (in its original steady state) is hit by a similar tropical cyclone, but also, at the same time, the world has learned that global warming implies a permanent increase in the depreciation rate of physical capital, $\delta_{CC} > \delta$. What does the new depreciation rate imply for long-run capital per efficiency unit, wage, rental rate, and consumption? Comment on the Golden Rule savings rate, s_{GR}, under the new climate-change-induced depreciation rate.

Solution:

A larger δ implies lower capital, which in turn implies lower wage and higher rental rate. Consumption falls, because output falls (and the savings rate is fixed). The Golden Rule steady state will see lower capital and lower consumption. However, the Golden Rule savings rate is unchanged.

(g) After the cyclone, the stricken economy observes that even though much of their installed capital was destroyed, capital per efficiency unit continues to fall rather than accumulate. Explain why this is happening.

Solution:

Capital will fall whenever it's above its long-run equilibrium level. The fact that the capital continues to fall implies that the amount of capital destroyed by the cyclone, was not sufficiently large to push the economy below its *new* long-run equilibrium, i.e. the steady state with δ_{CC}.