The exam consists of three parts, A, B, and C, with equal weight (1/3). Remember to allocate your time accordingly.

Part A (1/3 of the exam): Essay

Write a short essay addressing the following question in no more than 500 words. In addressing the question, relate to the course literature.

Solow (1997) wrote the following on the proper modelling strategy for short-run macroeconomic analysis:

> *My choices would be to model the main components of aggregate demand more or less opportunistically. By "opportunistically" I mean that whatever works (empirically). By "more or less" I mean I would want consumption functions, investment functions, import functions, and the like to look as if they could plausibly arise from aggregation of economic behavior of some reasonable kind at the micro level. That has always been the custom in macroeconomics, and I would not want to abandon it.*

Describe in what way a representative-agent model of consumption behavior does not "work" according to Solow's criterion. Discuss whether recent macroeconomic models ("HANK") satisfy Solow's "more or less opportunistic" approach, and in what way empirical work influences recent quantitative macroeconomic theory.

References

Solow, R. M. (1997). Is There a Core of Usable Macroeconomics We Should All Believe In? *American Economic Review*, 87(2):230–232.

Part B (1/3 of the exam): Investment Theory

Consider a project with a infinitely-lived cash flow d_{t+k} that is known in advance.

(a) Explain why the value V_0 of the project at time 0 with a constant interest rate is

$$V_0 = \sum_{t=1}^{\infty} \left(\frac{1}{1+r} \right)^t d_t. \tag{1}$$

(b) Suppose the cash flow is constant across time, $d_{t+k} = d$ for all $k > 0$. Show that the value V of the project is also constant and equal to $V = \frac{d}{r}$.

Consider a firm that solves the following problem

$$\max_{\{d_t, K_{t+1}, I_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \left(\frac{1}{1+r} \right)^t d_t$$

subject to

$$\begin{aligned} d_t &= K_t^{\alpha} - I_t, \\ K_{t+1} &= (1-\delta)K_t + I_t, \end{aligned}$$

where d is earnings, K is capital, I is investment, r is the interest rate, and δ is the depreciation rate.

(c) Show that the optimal level of capital is constant and equal to

$$K_t = K^* = \left(\frac{\alpha}{r+\delta} \right)^{\frac{1}{1-\alpha}} \quad \text{for all } t > 0 \tag{2}$$

(d) Explain the intuition for how an increase in the depreciation rate δ affects the level of capital.

(e) Show that the value of the firm is

$$V_0 = \frac{\left(\frac{\alpha}{r+\delta} \right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{\alpha}{r+\delta} \right)^{\frac{1}{1-\alpha}}}{r}. \tag{3}$$

(f) Explain intuitively through which channels a decrease in the interest rate r affects the value of the firm.

Part C (1/3 of the exam): Piketty

In his bestseller *Capital in the 21st century*, Thomas Piketty introduced what he termed the *second fundamental law of capitalism*:

$$\frac{K}{Y} = \frac{s}{g} \tag{4}$$

That is, the long-run capital-to-income ratio equals the savings rate divided by the growth rate of the economy. In this question, we will derive this law, explore its consequences, and compare it with what we covered in class.

Assume that capital accumulation is given by the following law of motion:

$$K_{t+1} = sY_t + K_t \tag{5}$$

where K_t is the capital stock at time t, Y_t is the income at time t, and s is the savings rate. That is, the capital stock tomorrow equals the capital stock today, K_t, plus investment today, sY_t.

(a) Assume further that the economy is on a balanced growth path such that $Y_{t+1} = (1+g)Y_t$ and $K_{t+1} = (1+g)K_t$. By manipulating Equation (5), show that the capital-to-income ratio K_t/Y_t satisfies the second fundamental law of capitalism, Equation (4).

(b) Piketty uses the second fundamental law of capitalism to forecast the long-run capital-to-income ratio if growth falls from $g = 0.02$ (2% growth rate) to $g = 0.01$ (1% growth rate), assuming that the savings rate remains constant. Describe, in at most three sentences, the consequences for the capital-to-income ratio of this change in growth rate.

(c) Equation (5) looks slightly different from what we covered in class. The capital accumulation equation in the Solow model was given by

$$K_{t+1} = sY_t + (1-\delta)K_t. \tag{6}$$

Explain the difference between Equation (5) and Equation (6). In particular, what does the parameter δ represent in the Solow model?

(d) Assume again that the economy is on a balanced growth path such that $Y_{t+1} = (1+g)Y_t$ and $K_{t+1} = (1+g)K_t$. By manipulating Equation (6), show that the capital-to-income ratio in the Solow model K_t/Y_t satisfies

$$\frac{K_t}{Y_t} = \frac{s}{g+\delta}.$$

(e) Assume that $\delta = 0.1$. Describe the effect of a fall in growth from $g = 0.02$ to $g = 0.01$ on the capital-to-income ratio in the Solow model, assuming that the savings rate remains constant. In particular, contrast with the forecast of Piketty's model.

(f) In the Solow model, define net income as $\hat{Y}_t = Y_t - \delta K_t$ and the net savings rate at time t as

$$\hat{s}_t = \frac{sY_t - \delta K_t}{\hat{Y}_t}.$$

Explain in words what net income and net savings rate are.

(g) Show that the Solow model can be rewritten on the net form

$$K_{t+1} = \hat{s}_t \hat{Y}_t + K_t.$$

The distinction between Piketty's model and the Solow model is thus a difference between whether we formulate our model in terms of net or gross income. However, they imply different theories of savings behavior. One predicts that the net savings rate is constant when growth falls, while the other predicts that the gross savings rate is constant when growth falls.

Solution Proposal

Part A

Here are criteria for answering part A well.

1. The student should demonstrate an understanding of some of the shortcomings of the representative-agent model of consumption behavior.

   (a) The representative-agent model assumes that aggregate consumption behavior can be described as the solution to an optimization problem of one single household.

   (b) This assumption does not work along several dimensions, the student can, for example, explain that it does not match empirical evidence on marginal propensities to consume.

2. The student should demonstrate an understanding of what heterogeneous-agent models are.

   (a) The models aim to aggregate "economic behavior of some reasonable kind at the micro level" and are based on forward-looking optimizing agents.

   (b) The student should list some arguments for and/or against HANK models satisfying Solow's criterion. For example, that heterogeneous-agent models can be made to match key micro facts such as the empirical marginal propensity to consume, but that they still rely on the assumption of rational expectations.

3. The student should demonstrate an understanding of the role of empirical work for quantitative macroeconomic theory.

   (a) This can for example be done by describing the work on estimating the marginal propensity to consume.

4. The essay should be well structured, well written and respond to the essay prompt.

If all four criteria are satisfied, the student should get a full score.

Part B

(a) The value of a project is the net present value of the cash-flow discounted using the constant interest rate r.

(b) If $d_t = d$ for all $t > 0$, then

$$V = \sum_{t=1}^{\infty} \left(\frac{1}{1+r}\right)^t d = d\frac{1}{1+r}\sum_{t=1}^{\infty}\left(\frac{1}{1+r}\right)^{t-1} = d\frac{1}{1+r}\frac{1}{1-\frac{1}{1+r}} = \frac{d}{r}.$$

(c) To solve this problem, one can set up the Lagrangian as

$$\mathcal{L} = \max_{\{d_t, K_{t+1}, I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (d_t - \lambda_t(d_t - K_t^{\alpha} + I_t) - q_t(K_{t+1} - (1-\delta)K_t - I_t))$$

with the corresponding first-order conditions

$$\partial\mathcal{L}/\partial d_t = \left(\frac{1}{1+r}\right)^t (1-\lambda_t) = 0, \qquad \partial\mathcal{L}/\partial K_{t+1} = \left(\frac{1}{1+r}\right)^t (-\lambda_t + q_t) = 0,$$

$$\partial\mathcal{L}/\partial I_t = -\left(\frac{1}{1+r}\right)^t q_t + \left(\frac{1}{1+r}\right)^{t+1}\left(\lambda_{t+1}\alpha K_{t+1}^{\alpha-1} + q_{t+1}(1-\delta)\right) = 0,$$

which implies that $\lambda_t = q_t = 1$ for all $t > 0$. The last first-order condition is then

$$-1 + \left(\frac{1}{1+r}\right)\left(\alpha K_{t+1}^{\alpha-1} + (1-\delta)\right) = 0$$

which becomes the solution proposed on the exam after some algebra.

(d) If δ increases, the level of capital goes down. The intuition is that if capital depreciates faster, any level of capital yields a return for a shorter period of time and is thus less valuable. The firm reacts by reducing the level of capital.

(e) To find the value of the firm, first note that since capital is constant, the earnings is also constant. Hence, the value is simply $V_0 = d/r$ as we found in problem (*b*). With constant capital, we know that $I = \delta K$ (using the law of motion for capital) and therefore that $d = K^{\alpha} - \delta K$. The value of the firm is then $V_0 = \frac{K^{\alpha}-\delta K}{r}$. If you insert the solution from problem (c) for K, you get the desired solution.

(f) A decrease in r affects the value of the firm through the following channels. First, a decrease in r affects the optimal level of capital, and thus earnings. A reduction in r reduces the required return on the project, inducing the firm to increase the level of capital when they face diminishing returns to scale. A higher level of capital increases the earnings flow d of the project (the student can solve $\partial d/\partial K$ to prove this, but it is not necessary), which affects the value of the firm positively. Second, a decrease in r affects the pricing of the earnings directly because it raises the present value of future cash flows.

Part C

(a) **Balanced growth with** $K_{t+1} = sY_t + K_t$. Assume a balanced growth path (BGP) with $Y_{t+1} = (1+g)Y_t$ and $K_{t+1} = (1+g)K_t$. Substituting $K_{t+1} = (1+g)K_t$ into the law of motion,

$$(1+g)K_t = sY_t + K_t \implies gK_t = sY_t \implies \frac{K_t}{Y_t} = \frac{s}{g}.$$

(b) **Effect of a fall in g from 0.02 to 0.01 with constant s.** Because $K/Y = s/g$, halving g doubles the long-run capital–income ratio. Quantitatively, K/Y rises by a factor $0.02/0.01 = 2$ (e.g., if $s = 0.10$, K/Y goes from 5 to 10).

(c) **Difference between $K_{t+1} = sY_t + K_t$ and the Solow equation $K_{t+1} = sY_t + (1-\delta)K_t$.** The Solow equation includes depreciation at rate $\delta \in (0,1)$: a fraction δ of the *gross* capital stock wears out each period. In contrast, $K_{t+1} = sY_t + K_t$ either (i) assumes $\delta = 0$ or (ii) treats Y_t as income net of replacement investment. (for full credit, it is sufficient to mention (i))

(d) **BGP ratio in the Solow model.** On a BGP, $K_{t+1} = (1+g)K_t$ and $Y_{t+1} = (1+g)Y_t$. Using $K_{t+1} = sY_t + (1-\delta)K_t$:

$$(1+g)K_t = sY_t + (1-\delta)K_t \implies (g+\delta)K_t = sY_t \implies \frac{K_t}{Y_t} = \frac{s}{g+\delta}.$$

(e) **Impact of lowering g from 0.02 to 0.01 when $\delta = 0.1$ (constant s).** In Solow, $K/Y = s/(g+\delta)$. The ratio rises from $s/0.12$ to $s/0.11$, i.e. by the factor $0.12/0.11 \approx 1.091$ (about a 9.1% increase). Unlike Piketty's net formulation (which predicts a doubling), depreciation in the denominator dampens the sensitivity of K/Y to a fall in g.

(f) **Net income and the net savings rate.** Define net income as $\hat{Y}_t \equiv Y_t - \delta K_t$, i.e. output net of capital consumption (replacement investment). The time-t *net* savings rate is

$$\hat{s}_t \equiv \frac{sY_t - \delta K_t}{\hat{Y}_t}.$$

Here sY_t is gross investment, δK_t is replacement (to keep K intact), so $sY_t - \delta K_t$ is *net* investment; dividing by net income gives the share of net income that is (net) saved.

(g) **Net formulation of the Solow model.** Starting from $K_{t+1} = sY_t + (1-\delta)K_t$, rewrite as

$$K_{t+1} = K_t + \underbrace{\left(sY_t - \delta K_t\right)}_{\text{net investment}}.$$

By the definition of $\hat{s}_t$,

$$sY_t - \delta K_t \;=\; \hat{s}_t\left(Y_t - \delta K_t\right) \;=\; \hat{s}_t\,\hat{Y}_t.$$

Hence,

$$K_{t+1} \;=\; \hat{s}_t\,(Y_t - \delta K_t) + K_t,$$

which is the desired *net* representation.