

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2400 — Real Analysis

Day of examination: Friday June 01. 2018.

Examination hours: 14.30–18.30

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All problems (1a, 1b, 2a, 2b etc.) count for 10 points each. You have to explain all answers, and show enough details so that it is easy to follow your arguments. At the end of this document you will find some definitions that might be handy. You may answer the exam in either English or Norwegian.

Problem 1

Let X be the space $X = C([0, 1], \mathbb{R})$ of continuous real valued functions on the interval $[0, 1]$. We equip X with the sup-norm, *i.e.*, for $f \in X$ we set

$$\|f\| = \sup_{x \in [0, 1]} \{|f(x)|\},$$

and we get the induced metric $d(f, g) = \|f - g\|$ for all $f, g \in X$.

(a) Let $L : X \rightarrow X$ be the map defined by

$$L(f)(x) = \int_0^x f(s)^3 ds.$$

Show that L has directional derivatives at each point $f \in X$.

(b) Show that L is differentiable at each point $f \in X$.

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Problem 2

We let X be the space from Problem 1, and we define a map $T : X \rightarrow X$ by

$$T(f)(x) = \int_0^x \cos\left(\frac{f(s)}{2}\right) ds.$$

- (a) Show that the function $g(x) = \cos(\frac{x}{2})$ is Lipschitz continuous on $\mathbb{R}$.
- (b) Show that T is a continuous mapping from X to X (We will take for granted that $T(f) \in X$ for each $f \in X$, so you don't have to show that.)
- (c) Show that $T : X \rightarrow X$ has a unique fixed point.

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Problem 3

Let Y be a non-empty set, and let d_1 and d_2 be metrics on Y . We let $d : Y \times Y \rightarrow \mathbb{R}$ be the function

$$d(x, y) = \sup\{d_1(x, y), d_2(x, y)\},$$

for all $x, y \in Y$.

- (a) Show that d is a metric.
- (b) Let $E = \{y_j\}_{j \in \mathbb{N}} \subset Y$ be a sequence of points, which is a Cauchy sequence with respect to both d_1 and d_2 . Show that E is a Cauchy sequence with respect to d .

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Problem 4

For $n \in \mathbb{N}$ let $f_n : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ denote the function

$$f_n(x) = \cos\left(\frac{\pi}{2}\left(1 - \frac{\pi - 2x}{\pi}\right)^n\right).$$

- (a) Show that f_n converges pointwise to a function f as $n \rightarrow \infty$.
- (b) Does f_n converge uniformly to f , *i.e.*, in sup-norm?

Problem 5

Let X be the space from Problem 1. Let $F : X \rightarrow \mathbb{R}$ be a continuous map, and assume that $F(f) = 0$ whenever $f \in X$ is a polynomial. Show that $F(f) = 0$ for all $f \in X$.

THE END

Some facts: Recall that if $L : X \rightarrow Y$ is a map between linear spaces, then the directional derivative $L'(f; r)$ at a point $f \in X$ in the direction $r \in X$ is given by

$$L'(f; r) = \lim_{t \rightarrow 0} \frac{L(f + tr) - L(f)}{t},$$

provided that the limit exists.

Recall that a function $g : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous if there exists a constant $K \geq 0$ such that $|g(x) - g(y)| \leq K \cdot |x - y|$ for all $x, y \in \mathbb{R}$.